

The Variational Origin of Motion by Gaussian Curvature

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Abstract. A variational formulation of an image analysis problem has the nice feature that it is often easier to predict the effect of minimizing a certain energy functional than to interpret the corresponding Euler-Lagrange equations. For example, the equations of motion for an active contour usually contains a mean curvature term, which we know will regularize the contour because mean curvature is the first variation of curve length, and shorter curves are typically smoother than longer ones. In some applications it may be worth considering Gaussian curvature as a regularizing term instead of mean curvature. The present paper provides a variational principle for this: We show that Gaussian curvature of a regular surface in three-dimensional Euclidean space is the first variation of an energy functional defined on the surface. Some properties of the corresponding *motion by Gaussian curvature* are pointed out, and a simple example is given, where minimization of this functional yields a nontrivial solution.

Keywords: Total mean curvature, Gaussian curvature, gradient descent flow, level set methods, Euler characteristic.

1 Introduction

For almost two decades, following the publication of the seminal papers by Kass, Witkin, and Terzopoulos [10] and Mumford and Shah [13], variational principles have been both popular and powerful tools in the inventory of the image analysts' toolbox. The level set method of Osher and Sethian [15] has made it considerably easier to implement and visualize curve and surface evolutions such as *geometric active contours* [2], *geodesic active contours* [3], *active contours without edges* [4], and notably *motion by mean curvature* (MMC), see e.g. Brakke [1]. Existence and uniqueness of viscosity solutions of the level set equations for MMC was established simultaneously by Chen, Giga, and Goto [5] and Evans and Spruck [7].

The present paper focuses on *motion by Gaussian curvature* (MGC). By MGC we mean a differentiable one-parameter family of regular surfaces $I \ni t \mapsto \Gamma(t) \subset \mathbf{R}^3$ in three-dimensional Euclidean space, I being an open interval containing $t = 0$, which solves the initial value problem,

$$\frac{d}{dt}\Gamma(t) = -K_{\Gamma(t)} \quad , \quad \Gamma(0) = \Gamma_0 \quad , \quad (1)$$

for some given initial surface Γ_0 . Here $(d/dt)\Gamma(t)$ is the (scalar) normal velocity of the evolving surface, and $K = K_{\Gamma(t)} = K_{\Gamma(t)}(\mathbf{x})$ denotes the Gaussian curvature at $\mathbf{x} \in \Gamma(t)$.

Motion by Gaussian curvature has not received nearly as much attention as MMC. One of the first papers on the subject is that of Firey [8] who constructed an idealized model of the wearing process of a convex stone on the beach, assuming that the local rate of wear is proportional to the Gaussian curvature. Oliker [14] studied the MGC for a surfaces which are graphs, $\Gamma(t) : z = u(x, y, t)$, where the function $u : \bar{U} \times [0, \infty) \rightarrow \mathbf{R}$ is defined on bounded, strictly convex subset $\bar{U} \subset \mathbf{R}^2$ with smooth boundary ∂U . Since the Gaussian curvature of such a surface is given by, see do Carmo [6, p. 163],

$$K = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2}, \quad (2)$$

and the normal velocity of $t \mapsto \Gamma(t)$ is $(d/dt)\Gamma(t) = u_t/(1 + u_x^2 + u_y^2)^{1/2}$, substitution into (1) gives the PDE

$$u_t = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^{3/2}} \quad \text{in } U \times [0, \infty),$$

which is solved with homogeneous Dirichlet boundary conditions on ∂U .

In image analysis MGC has been used only in a few cases, most recently by Lee and Seo [11]. One way to understand this lack of use is to observe that MMC enters into the existing variational segmentation models as a *regularizing term* alongside the *fidelity term*. This application is based on a variational principle, namely that MMC is the gradient descent motion for the minimization of the surface area functional

$$E_A(\Gamma) = \int_{\Gamma} dA, \quad ,$$

where dA is the element of surface area on Γ . A typical segmentation functional has the form $E = E_F + \lambda E_A$, where $\lambda > 0$ is a parameter. The first term E_F is the fidelity term, which contains all the information about the input image, and the second is the area term λE_A , which is included as a smoothness prior. Smoothness is achieved as a trade-off between a good fit of the model to the input image, on one hand, and a small surface area of the interface between object and background, on the other. When gradient descent minimization is applied to E , the resulting evolution equation contains a mean curvature term. The variational interpretation of MMC as a minimizing flow of the surface area enables us to predict the regularizing nature of this mean curvature term. For MGC the corresponding variational interpretation is not well-known, making it harder to see the effects of including a Gaussian curvature term into an evolution equation. This may be one of the reasons why MGC has not been used so much.

In this paper we show that MGC is the gradient descent evolution for the minimization of an energy functional defined on the evolving surface. In fact, if

we consider the *total mean curvature* of a regular surface Γ in \mathbf{R}^3 ,

$$E_{\mathbf{H}}(\Gamma) := \int_{\Gamma} H \, dA , \quad (3)$$

where $H = H(\mathbf{x})$ denotes the mean curvature of the surface at $\mathbf{x} \in \Gamma$, then we prove that the *first variation* (or Gâteaux derivative or directional derivative) of $E_{\mathbf{H}}$ is

$$dE_{\mathbf{H}}(\Gamma)(v) = \int_{\Gamma} K v \, dA , \quad (4)$$

for all normal variations $v : \Gamma \rightarrow \mathbf{R}$ of the surface. Loosely speaking, Gaussian curvature K is the first variation of the total mean curvature.

One way of proving (4) is to use that any regular surface may locally be considered as the graph $\Gamma = \Gamma(u) : z = u(x, y)$ of a smooth function $u : U \rightarrow \mathbf{R}$ defined on some bounded, open subset U of \mathbf{R}^2 . In this representation the mean curvature of the surface is given by [6, p. 163],¹

$$H = -\frac{1}{2} \frac{(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}}{(1 + u_x^2 + u_y^2)^{3/2}} .$$

Since the area element on $\Gamma(u)$ is $dA = (1 + u_x^2 + u_y^2)^{1/2} \, dx dy$, the total mean curvature becomes a functional in u of the form,

$$E(u) := E_{\mathbf{H}}(\Gamma(u)) = - \int_U \frac{(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}}{2(1 + u_x^2 + u_y^2)} \, dx dy .$$

If $\varphi \in C_0^\infty(U)$ is a test function, then $t \mapsto u + t\varphi$ is a variation of u which corresponds to a local smooth deformation of the surface Γ . The first variation of the total mean curvature thus becomes

$$\begin{aligned} dE(u)(\varphi) &= \left. \frac{d}{dt} E(u + t\varphi) \right|_{t=0} \\ &= \int_U \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2} \varphi \, dx dy = \int_{\Gamma} K v \, dA , \end{aligned}$$

where $v = \varphi/(1 + u_x^2 + u_y^2)^{1/2}$ is the normal variation of $t \mapsto \Gamma(u + t\varphi)$ at $t = 0$. We have also used the formula (2) for the Gauss curvature K , and the expression for the area element dA on $\Gamma(u)$. This identity proves (4).

This proof is both straight-forward and reliable, but at the same time tedious and dull, because of the long routine calculations involved. The method of proof that we are going to propose below is more geometrical in nature, and applies to more general situations without extra work. In fact, we are going to formulate a somewhat more general result, Theorem 1 in Sect. 3, which has (4) as a simple

¹ Notice our sign convention: The sphere of radius R has positive mean curvature $H = 1/R$. In [6] and [12] the opposite sign is used.

corollary.² After some geometrical preliminaries in Sect. 4, the proof is sketched in Sect. 5. Motion by Gaussian curvature is considered briefly in Sect. 6, and finally, surfaces of revolution with minimal total mean curvature in Sect. 7.

2 Volume, Surface Area, and Total Mean Curvature

Let Γ denote a compact regular surface in \mathbf{R}^3 . Notice that Γ 's complement consists of exactly two components, one of which is bounded. This bounded component is called the *inside of Γ* , and will be denoted Ω . Let $w \in C^\infty(\mathbf{R}^3)$ be an arbitrary smooth function, and consider the following weighted surface functionals: First of all the **weighted volume**,

$$E_V(w, \Gamma) := \int_{\Omega} w \, d\mathbf{x} \quad , \quad (5)$$

then the **weighted surface area**,

$$E_A(w, \Gamma) := \int_{\Gamma} w \, dA \quad , \quad (6)$$

and the **weighted total mean curvature**,

$$E_H(w, \Gamma) := \int_{\Gamma} Hw \, dA \quad . \quad (7)$$

Finally, we also consider the **weighted total Gaussian curvature**,

$$E_K(w, \Gamma) := \int_{\Gamma} Kw \, dA \quad . \quad (8)$$

Notice that the map $C^\infty(\mathbf{R}^3) \ni w \mapsto E_A(w, \Gamma) \in \mathbf{R}$ defines a (Schwartz-) distribution with compact support, that is $E_A(\cdot, \Gamma) \in \mathcal{E}'(\mathbf{R}^3)$. On the other hand, $\Gamma \mapsto E_A(w, \Gamma)$ defines a surface functional, in the usual sense. This holds for the other functionals E_V , E_H , and E_K , as well. If $w \equiv 1$ we write

$$\begin{aligned} E_V(\Gamma) &:= E_V(1, \Gamma), & E_A(\Gamma) &:= E_A(1, \Gamma) \quad , \\ E_H(\Gamma) &:= E_H(1, \Gamma), & \text{and} & \quad E_K(\Gamma) := E_K(1, \Gamma) \quad , \end{aligned}$$

corresponding to the volume of Ω , the surface area of Γ , and total mean and Gaussian curvatures of Γ , respectively. One of the most famous results in classical global differential geometry, namely *Gauss-Bonnet's Theorem*, tells us that the value of the total Gaussian curvature $E_K(\Gamma)$ is entirely determined by the topological type of Γ ,

$$E_K(\Gamma) = \int_{\Gamma} K \, d\sigma = 2\pi\chi(\Gamma) \quad , \quad (9)$$

² It has been brought to our attention that the results in (4) and Theorem 1 may be found in a more general version in [9, 1.6 Scholia, p. 82]. However, since the results do not seem to be commonly known, and our proof is new and simple, we believe this paper is still of interest to members of the image analysis community.

where $\chi(\Gamma)$ denotes the *Euler characteristic* of the surface Γ , i.e. $\chi(\Gamma) = 2 - 2g$, where g is the *genus* of the surface, cf. [6, p.273].

3 The First Variation of Total Mean Curvature

The first variation, or Gâteaux derivative, of a surface functional $E = E(\Gamma)$ is a mapping $v \mapsto dE(\Gamma)(v)$ defined by the derivative

$$dE(\Gamma)(v) = \left. \frac{d}{dt} E(\Gamma(t)) \right|_{t=0} ,$$

where $t \mapsto \Gamma(t)$ is an evolving surface satisfying $\Gamma(0) = \Gamma$ and $(d/dt)\Gamma(t) = v$. The latter means that the *normal velocity*, or *normal variation*, of the surface evolution is given by the function $v : \Gamma \rightarrow \mathbf{R}$. The first variation of a surface functional is homogeneous of degree one, by definition, but not necessarily additive, hence generally not a linear mapping. If the first variation dE happens to be linear, then we call it the *differential of E*. Two such differentials, which are extensively used in image analysis, are that of the weighted volume,

$$dE_V(w, \Gamma)(v) = \int_{\Gamma} wv \, dA = E_A(wv, \Gamma) , \quad (10)$$

and weighted surface area,

$$dE_A(w, \Gamma)(v) = \int_{\Gamma} w_n v \, dA + 2 \int_{\Gamma} H w v \, dA = E_A(w_n v, \Gamma) + 2E_H(wv, \Gamma), \quad (11)$$

where w_n denotes the normal derivative on Γ of the function $w \in C^\infty(\mathbf{R}^3)$, and v is any normal variation of the surface. Readers will recognize (11) as the differential of the *geodesic active contours* [3]. Missing from the above list is the first variations of the total mean curvature functional E_H and the Gaussian curvature functional E_K . They are provided by the main result of this paper:

Theorem 1. *Let Γ be a compact regular surface in \mathbf{R}^3 and $w \in C^\infty(\mathbf{R}^3)$. The first variations of the weighted total mean curvature (7) and the weighted Gaussian curvature (8) are given by*

$$dE_H(w, \Gamma)(v) = E_H(w_n v, \Gamma) + E_K(wv, \Gamma) , \quad (12)$$

and

$$dE_K(w, \Gamma)(v) = E_K(w_n v, \Gamma) , \quad (13)$$

for any normal variation $v : \Gamma \rightarrow \mathbf{R}$, and w_n denoting the normal derivative of w . Both variations are linear functionals of v , hence they are the differentials of E_H and E_K , respectively.

If w is identically equal to one then, as an easy corollary of the theorem, we find that $dE_H(\Gamma)(v) = E_K(v, \Gamma)$ and $dE_K(\Gamma)(v) = 0$. The first identity is exactly the assertion in (4), and the second is a consequence of Gauss-Bonnet's theorem (9).

4 Some Geometric Preliminaries

We prepare for the proof of Theorem 1 in the next section by recalling some facts from differential geometry. Suppose Γ is a compact regular surface in \mathbf{R}^3 , \mathbf{x}_0 a point on Γ , and let $\mathbf{x} = \mathbf{x}(u, v)$ be a local parametrization of a neighbourhood of \mathbf{x}_0 , with parameters $(u, v) \in U$ where $U \subset \mathbf{R}^2$ is an open set. In the parameterized patch $\mathbf{x}(U)$ the Euclidean surface area element dA is given by the formula,

$$dA = |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv \quad , \quad (14)$$

where \mathbf{x}_u and \mathbf{x}_v are the partial derivatives of \mathbf{x} with respect to u and v , respectively, and “ \wedge ” denotes the vector product in \mathbf{R}^3 . The principal curvatures of the surface at a point \mathbf{x} on Γ are denoted $\kappa_1 = \kappa_1(\mathbf{x})$ and $\kappa_2 = \kappa_2(\mathbf{x})$, respectively. The *mean curvature* $H = H(\mathbf{x})$ and the *Gaussian curvature* $K = K(\mathbf{x})$ are then given by

$$H = \frac{\kappa_1 + \kappa_2}{2} \quad \text{and} \quad K = \kappa_1 \kappa_2 \quad .$$

Denote by $\mathbf{n} = \mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \Gamma$, the *outward unit normal* on Γ (Recall that we have a well-defined “inside”.) The principal curvatures at \mathbf{x} are the eigenvalues of the differential $D\mathbf{n}(\mathbf{x})$ which maps the tangent space at \mathbf{x} into itself. The surface Γ is said to be locally parameterized by *lines of principal curvature* if the parametrization $\mathbf{x} = \mathbf{x}(u, v) : U \rightarrow \mathbf{R}^3$ satisfies

$$\mathbf{n}_u(u, v) = \kappa_1(u, v)\mathbf{x}_u(u, v) \quad \text{and} \quad \mathbf{n}_v(u, v) = \kappa_2(u, v)\mathbf{x}_v(u, v) \quad , \quad (15)$$

that is, the coordinate directions \mathbf{x}_u and \mathbf{x}_v are eigenvectors of the differential $D\mathbf{n}(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}(u, v)$.

For $t \in \mathbf{R}$ define $\Gamma(t) = \{\mathbf{x} \in \mathbf{R}^3 : d(\mathbf{x}, \Gamma) = t\}$, where $d(\cdot, \Gamma)$ is the signed distance to the surface Γ . Since Γ is assumed to be compact there exists a real number $\epsilon > 0$ such that if $t \in (-\epsilon, \epsilon)$, then the set $\Gamma(t)$ is again a compact regular surface, called a *parallel surface* to Γ . If $\mathbf{x} = \mathbf{x}(u, v)$ is a local parametrization of Γ , then each member the one-parameter family of parallel surfaces $t \mapsto \Gamma(t)$, $t \in (-\epsilon, \epsilon)$, can be parameterized locally by

$$\mathbf{x}^t = \mathbf{x}^t(u, v) := \mathbf{x}(u, v) + t\mathbf{n}(u, v) \quad . \quad (16)$$

Notice that $t \mapsto \Gamma(t)$ is the surface evolution satisfying the initial value problem,

$$\frac{d}{dt}\Gamma(t) = 1 \quad \text{on} \quad \Gamma(t), \quad \text{and} \quad \Gamma(0) = \Gamma \quad . \quad (17)$$

The area element on the parallel surface $\Gamma(t)$ can be expressed in terms of the area element on Γ and its curvatures:

Lemma 1. *The Euclidean area element dA^t on the parallel surface $\Gamma(t)$ with the local parametrization (16) is given by*

$$dA^t = (1 + 2tH + t^2K) dA, \quad (-\epsilon < t < \epsilon) \quad ,$$

where $dA = dA^0$ is the area element on Γ .

Proof. We prove the lemma under the simplifying assumption that the local parametrization can be chosen such that the coordinate lines $u \mapsto \mathbf{x}(u, v)$ and $v \mapsto \mathbf{x}(u, v)$ are *lines of principal curvature*, in which case it follows that

$$\begin{aligned} \mathbf{x}_u^t(u, v) &= \mathbf{x}_u(u, v) + t\mathbf{n}_u(u, v) \\ &= \mathbf{x}_u(u, v) + t\kappa_1(u, v)\mathbf{x}_u(u, v) = (1 + t\kappa_1(u, v))\mathbf{x}_u(u, v) , \end{aligned}$$

and similarly $\mathbf{x}_v^t(u, v) = (1 + t\kappa_2(u, v))\mathbf{x}_v(u, v)$. Using (14) we find that

$$\begin{aligned} dA^t &= |\mathbf{x}_u^t \wedge \mathbf{x}_v^t| dudv \\ &= |(1 + t\kappa_1)\mathbf{x}_u \wedge (1 + t\kappa_2)\mathbf{x}_v| dudv \\ &= (1 + t\kappa_1)(1 + t\kappa_2)|\mathbf{x}_u \wedge \mathbf{x}_v| dudv = (1 + t(\kappa_1 + \kappa_2) + t^2\kappa_1\kappa_2) dA , \end{aligned}$$

which is the desired result. For a complete proof, see [12, p. 145]. \square

A point \mathbf{x} whose distance to Γ is less than ϵ is said to belong to a tubular neighbourhood T_ϵ of Γ . Any $\mathbf{x} \in T_\epsilon$ has a unique representation of the form $\mathbf{x} = \mathbf{x}_0 + t\mathbf{n}(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in \Gamma$ and some $t \in (-\epsilon, \epsilon)$. The point \mathbf{x}_0 is called \mathbf{x} 's *projection onto Γ* , and t is the *signed distance* of \mathbf{x} to Γ . This representation is used in the proof of the following result:

Corollary 1. *For any weight $w \in C^\infty(\mathbf{R}^3)$, and the one-parameter family of parallel surfaces $t \mapsto \Gamma(t)$ defined by (16), we have the Taylor expansion,*

$$\begin{aligned} E_A(w, \Gamma(t)) &= E_A(w, \Gamma) + t(E_A(w_n, \Gamma) + 2E_H(w, \Gamma)) + \\ &\quad + \frac{1}{2}t^2[E_A(w_{nn}, \Gamma) + 4E_H(w_n, \Gamma) + 2E_K(w, \Gamma)] + O(t^3) , \end{aligned}$$

as $t \rightarrow 0$, where w_n and w_{nn} denote the first and second derivatives of w in the direction normal to the surface Γ .

Proof. For each point $\mathbf{x} \in \Gamma$ fixed, the function $t \mapsto w(\mathbf{x} + t\mathbf{n}(\mathbf{x}))$ has the following Taylor expansion,

$$w(\mathbf{x} + t\mathbf{n}(\mathbf{x})) = w(\mathbf{x}) + tw_n(\mathbf{x}) + \frac{1}{2}t^2w_{nn}(\mathbf{x}) + O(t^3) ,$$

which, in combination with the formula for dA^t in the Lemma 1, gives the desired result. \square

Any smooth function $v : \Gamma \rightarrow \mathbf{R}$ has a smooth extension to a tubular neighbourhood T_ϵ of Γ which is constant along rays normal to the surface. This extension, which is also denoted v , is given by the formula

$$v(\mathbf{x}) = v(\mathbf{x}_0), \quad (\mathbf{x} \in T_\epsilon) \tag{18}$$

where \mathbf{x}_0 is the unique projection of \mathbf{x} onto Γ . This extension is convenient in the formulation of the lemma below, and will play an important role in the proof of Theorem 1.

Let $s \mapsto \Gamma(s)$ be a surface evolution defined for $s \in I$, where I is an open interval containing $s = 0$. For $s \in I$ fixed, let $t \mapsto \Gamma(s)(t) := \Gamma(s, t)$ denote the family of parallel surfaces of $\Gamma(s)$. Then we have,

Lemma 2. *If the normal velocity of $s \mapsto \Gamma(s)$ at $s = 0$ is given by the scalar function $(d/ds)\Gamma(s)|_{s=0} = v$, then, for t fixed, the normal velocity at $s = 0$ of the evolution $s \mapsto \Gamma(s, t)$ of a parallel surface is*

$$\left. \frac{d}{ds} \Gamma(s, t; \mathbf{x}) \right|_{s=0} = v(\mathbf{x}) \quad (\text{for } \mathbf{x} \in \Gamma(0, t)) ,$$

where v is the extension (18) of the normal velocity $v : \Gamma(0) \rightarrow \mathbf{R}$ to a tubular neighbourhood of $\Gamma(0)$.

Proof. The proof is carried out in a local parametrization $\mathbf{x} = \mathbf{x}(u, v, s)$ of the evolution $s \mapsto \Gamma(s)$. The corresponding parametrization of the parallel surfaces $s \mapsto \Gamma(s, t)$ is then given by

$$\mathbf{x}^t = \mathbf{x}^t(u, v, s) = \mathbf{x}(u, v, s) + t\mathbf{n}(u, v, s) ,$$

where $\mathbf{n} = \mathbf{n}(u, v, s)$ is the parametrization of the outward unit normal of $\Gamma(s)$. Using the notation $\dot{} = d/ds$ we find that

$$\begin{aligned} \left. \frac{d}{ds} \Gamma(s, t; \mathbf{x}^t) \right|_{s=0} &= \mathbf{n}(u, v, 0) \cdot \dot{\mathbf{x}}^t(u, v, 0) \\ &= \mathbf{n}(u, v, 0) \cdot \dot{\mathbf{x}}(u, v, 0) + t\mathbf{n}(u, v, 0) \cdot \dot{\mathbf{n}}(u, v, 0) \\ &= \mathbf{n}(u, v, 0) \cdot \dot{\mathbf{x}}(u, v, 0) = \left. \frac{d}{ds} \Gamma(s; \mathbf{x}) \right|_{s=0} = v(\mathbf{x}) , \end{aligned}$$

because $0 = (d/ds)|\mathbf{n}(u, v, s)|^2 = 2\mathbf{n}(u, v, s) \cdot \dot{\mathbf{n}}(u, v, s)$ for all (u, v, s) . This proves the lemma because \mathbf{x} is the projection of \mathbf{x}^t onto $\Gamma(0)$. \square

5 Proof of the Main Theorem

We now come to the proof of Theorem 1 itself. Again, let $t \rightarrow \Gamma(t)$ denote the family of surfaces parallel to Γ , and observe that equations (10) and (17) imply that

$$\frac{d}{dt} E_V(w, \Gamma(t)) = dE_V(w, \Gamma(t)) \left(\frac{d}{dt} \Gamma(t) \right) = dE_V(w, \Gamma(t))(1) = E_A(w, \Gamma(t)) .$$

The right hand side of this identity is known from Corollary 1, so by integrating we find the following Taylor expansion of the weighted volume functional on the parallel surface $\Gamma(t)$ as $t \rightarrow 0$:

$$E_V(w, \Gamma(t)) = E_V(w, \Gamma) + tE_A(w, \Gamma) + \frac{1}{2}t^2 [E_A(w_n, \Gamma) + 2E_H(w, \Gamma)] + O(t^3) . \tag{19}$$

Now, the idea is to use the fact that the (19) holds for any surface Γ and its parallel surface $\Gamma(t)$, for any fixed sufficiently small t . We begin by computing the differential with respect to normal variations v of Γ on both sides of the

equality sign. Using Lemma 2 to find the normal variation of $\Gamma(t)$ in terms v in the left hand side of (19) it follows that,

$$\begin{aligned} dE_V(w, \Gamma(t))(v) &= dE_V(w, \Gamma)(v) + tdE_A(w, \Gamma)(v) + \\ &+ \frac{1}{2}t^2(dE_A(w_n, \Gamma)(v) + 2dE_H(w, \Gamma)(v)) + O(t^3) . \end{aligned}$$

Substituting the formulas for dE_V and dE_A in (10) and (11) into this identity gives

$$\begin{aligned} E_A(wv, \Gamma(t)) &= E_A(wv, \Gamma) + t[E_A(w_nv, \Gamma) + 2E_H(wv, \Gamma)] + \\ &+ \frac{1}{2}t^2[E_A(w_{nn}v, \Gamma) + 2E_H(w_nv, \Gamma) + 2dE_H(w, \Gamma)(v)] + O(t^3). \end{aligned} \quad (20)$$

Now, replace the test function w by in Corollary 1 by the product wv , where v is the extension of the normal velocity $v : \Gamma \rightarrow \mathbf{R}$ defined in (18). Since v is constant along rays normal to the surface, $(wv)_n = w_nv$ and $(wv)_{nn} = w_{nn}v$, so we get

$$\begin{aligned} E_A(wv, \Gamma(t)) &= E_A(wv, \Gamma) + t[E_A(w_nv, \Gamma) + 2E_H(wv, \Gamma)] + \\ &+ \frac{1}{2}t^2[E_A(w_{nn}v, \Gamma) + 4E_H(w_nv, \Gamma) + 2E_K(wv, \Gamma)] + O(t^3) , \end{aligned}$$

If we compare the coefficients in this expansion with those found in the Taylor expansion (20) we find that

$$dE_H(w, \Gamma)(v) = E_H(w_nv, \Gamma) + E_K(wv, \Gamma) ,$$

which is the desired formula for the differential of the weighted total mean curvature. The differential for the weighted total Gaussian curvature can be obtained in a similar manner by including third order terms in the expansions. The details are left to the reader. \square

6 Some Properties of Motion by Gaussian Curvature

In this section we want to point to some interesting properties of the motion by Gaussian curvature, $t \mapsto \Gamma(t)$, defined by the initial value problem (1). Consider the volume of the interior $\Omega(t)$ of a surface $\Gamma(t)$,

$$V(t) := E_V(\Gamma(t)) = \int_{\Omega(t)} dx .$$

It follows from (10) with $w \equiv 1$, and the definition (1) of MGC, that

$$V'(t) = dE_V(\Gamma(t), \frac{d}{dt}\Gamma(t)) = dE_V(\Gamma(t), -K_{\Gamma(t)}) = -E_K(\Gamma(t)) , \quad (21)$$

so, in view of Gauss-Bonnet's theorem (9), we find the following differential equation

$$V'(t) = -2\pi\chi(\Gamma(t)) , \quad (22)$$

where $\chi(\Gamma)$ is the Euler characteristic of Γ . This equation has some interesting consequences. First of all, (22) seems to suggest that the surface does not change topological type as it evolves. This is true as long as $V(t)$ is continuously differentiable because the Euler characteristic is an integer, so a change of topological type would lead to a jump in the right hand side of the equation. Secondly, (22) shows that the volume of $\Omega(t)$ changes at a constant rate. For instance, if Γ_0 is homeomorphic to the two-sphere S^2 , then so is $\Gamma(t)$ for all sufficiently small $t > 0$, and since $\chi(S^2) = 2$, cf. [6, p. 273], it follows that

$$V'(t) = -4\pi \quad (\text{for } \Gamma_0 \text{ homeomorphic to } S^2).$$

In particular $\Gamma(t)$ ceases to exist after a certain *extinction time* t_* given by

$$t_* = \frac{V(0)}{4\pi} .$$

If Γ_0 is homeomorphic to the standard torus T^2 , then $\chi(\Gamma_0) = 0$ ([6, p. 273]), implying that

$$V(t) = V(0) \quad (\text{for } \Gamma_0 \text{ homeomorphic to } T^2),$$

that is, MGC preserves the volume of $\Omega(t)$. Finally, if Γ_0 is a surface of higher genus than the sphere or the torus (i.e. $g \geq 2$), then $\chi(\Gamma_0) < 0$, and we conclude that the volume $V(t)$ increases at a constant rate.

In Fig. 1 a comparison between MMC and MGC is shown for T^2 . MMC decreases surface area and leads to shrinking in contrast to MGC which does not change the volume but moves the surface closer to the symmetry axis.

7 Surfaces of Revolution with Minimal Total Mean Curvature

Let $u : [a, b] \rightarrow \mathbf{R}$ be a twice continuously differentiable function, such that $y(x) > 0$ for all $x \in [a, b]$, and $\Gamma(u) = \{(x, y, z) \in \mathbf{R}^3 \mid y^2 + z^2 = u(x)^2\}$ be the surface of revolution obtained by rotating u 's graph through an angle of 360° about the x -axis. We are now going to determine the surfaces of revolution which minimizes total mean curvature. The mean curvature of $\Gamma(u)$ is given by the formula

$$H = \frac{1 + (u')^2 - uu''}{2u(1 + (u')^2)^{3/2}} ,$$

and the surface area element by $dA = 2\pi u(1 + (u')^2)^{1/2} dx$, so the total mean curvature of $\Gamma(u)$ is the functional of u given by the formula,

$$E_H(u) := E_H(\Gamma(u)) = \pi \int_a^b \frac{1 + (u')^2 - uu''}{1 + (u')^2} dx . \quad (23)$$



Fig. 1. From left to right, a comparison between motion by mean curvature (top) and motion by Gaussian curvature (bottom) for the standard torus T^2 .

Let $A, B > 0$ and set

$$\mathcal{A} = \{u \in C^2([a, b]) \mid u(a) = A, u(b) = B \text{ and } u(x) > 0 \text{ for all } a < x < b\}.$$

The task is to find an admissible function $u = u_0$ such that

$$u_0 \in \mathcal{A} : \quad E_H(u_0) \leq E_H(u) \quad \text{for all } u \in \mathcal{A}.$$

Assume that such a function u_0 exists, and pick a test function $\varphi \in C_0^2(a, b)$. If $\epsilon > 0$ is sufficiently small, then the function $u_0(x) + t\varphi(x) \in \mathcal{A}$ for all $t \in (-\epsilon, \epsilon)$. The necessary condition for a minimum is,

$$0 = \frac{d}{dt} E_H(u_0 + t\varphi) \Big|_{t=0} = 2\pi \int_a^b \frac{-u_0''}{(1 + (u_0')^2)^2} \varphi \, dx .$$

The right hand side was obtained by differentiation with respect to t under the integral sign, followed by integration by parts, and some simplifications. Since the test function φ is arbitrary, the minimizer u_0 must satisfy $u_0''(x) = 0$ for all $x \in (a, b)$, hence

$$u_0(x) = (A(b - x) + B(x - a))/(b - a)$$

which is the straight line segment connecting the fixed endpoints (a, A) and (b, B) . The corresponding surface of revolution $\Gamma(u_0)$ is therefore a part of a circular cone. Whether the solution is a local minimum or just a stationary point is at present not known to us.

Although this example is simple, it shows that minimization problems for the total mean curvature, in the presence of boundary conditions or constraints, may yield nontrivial and meaningful results.

8 Conclusion

We have seen that motion by Gaussian curvature is the gradient descent flow for a geometric surface functional, namely the total mean curvature of the surface. This functional can be used in applications as an alternative to the area functional which leads to the frequently used mean curvature motion. Some properties of the Gaussian curvature motion were pointed out and minimization of the total mean curvature functional subject to boundary conditions was considered briefly in a simple case. More work remains to be done in the area and it will be interesting to see if the theory for motion by Gaussian curvature will become as rich as the one for motion by mean curvature.

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