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LETTER TO THE EDITOR

Comments on Marken and Shaffer: The power law of movement: an example of a behavioral illusion

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Abstract
Many researchers who have studied movements along curved paths, under a variety of conditions, by different organisms, mostly human but a couple with non-human organisms, have found a consistent form of relation between the tangential (along-track) instantaneous velocity V and the local radius of curvature R. The consistent relation is that $V \approx \mathrm{cR}^k$, where k is a constant less than unity, often near 0.33 but sometimes far from 0.33, and c is a proportionality constant appropriate to the organism and the situation (see Zago, Matic, Flash, et al. (2017) for many examples in which the power law holds with widely varying values of the power, as well as cases of simple systems for which everything can be calculated exactly and in which the power law fails badly). Marken and Shaffer (Exp Brain Res 235:1835–1842; 2017), following a challenge by Gomez-Marin to see whether it is possible to use Perceptual Control Theory (Powers 1973/2005) to explain the power law results (Alex Gomez-Marin posting to CSGnet@lists.illinois.edu 2016.05.03), claim to have found a mathematical argument that proves the true exponent of the power relating velocity and radius of curvature always to be 1/3. They say that deviations from this value occur because researchers have omitted a critical correction "cross-product" factor that the authors label "*D*". This note questions the logic of the analysis offered by Marken and Shaffer, and argues that even had the analysis been correct, it would not affect future research into the reasons why and when the power law is observed and the circumstances that determine the value of the power found when it is observed.

Mathematical background

On the face of it, it would be remarkable, even impossible, to derive a velocity directly from a length measure (radius of curvature). Have Marken and Shaffer managed to do the impossible, or have they erred in their analysis? I argue that their derivation of velocities directly from shapes is faulty, an error produced by treating one instance of a large set as being the only possible member of the set, as in "*I heard my friend got a dog she called "Casper". I saw a dog on the street today. It must have been called Casper*." In the present case, the instance is a measured velocity profile over a curved path, while the set consists of all the possible velocity profiles that could occur over that path.

Here is the mathematical background for the mistake. A curve on a plane surface can be described by selecting an arbitrary point on the curve and reporting the x and y values of other points as functions of distance "s" along the curve from that arbitrary point. The values of x and y are independent functions of distance along the curve, linked only by the actual shape of the curve. These two functions, $x = x(s)$ and $y = y(s)$, produce what is called a "parametric representation" of the curve with s as the parameter. The local (signed) radius of curvature at any point on the curve depends on the first and second derivatives of x and y with respect to s at that point according to a well-known expression:

$$R = \frac{\left(\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2\right)^{3/2}}{\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}^2x}{\mathrm{d}s^2}} \quad (1)$$

The parameter need not be the along-curve distance "s" itself. Any arbitrary continuous variable, say "z", may be used to specify the distance along the curve, if its value can always be converted unambiguously into a specific value of s. If $s = f(z)$ has that property, then z can be substituted directly for s in Eq. (1).

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To see this, note that $\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}z}$ and $\frac{\mathrm{d}^2x}{\mathrm{d}z^2} = \frac{\mathrm{d}^2x}{\mathrm{d}s^2}\left(\frac{\mathrm{d}s}{\mathrm{d}z}\right)^2 + \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}^2s}{\mathrm{d}z^2}$. Hence,

$$R = \frac{\left(\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2\right)^{3/2}}{\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}^2x}{\mathrm{d}s^2}} = \frac{\left(\left(\frac{\mathrm{d}x}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2\right)^{3/2}}{\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^3\left(\frac{\mathrm{d}x}{\mathrm{d}z}\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}^2x}{\mathrm{d}z^2}\right)} = \frac{\left(\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2\right)^{3/2}}{\frac{\mathrm{d}x}{\mathrm{d}z}\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}^2x}{dz^2}} \tag{2}$$

the last of which is just (1) with "z" replacing "s".

Of course, z could be any variable, including t (time) before or after some arbitrary starting moment when the moving indicator is at some point on the curve that specifies $s=0$, in which case $\mathrm{d}s/\mathrm{d}t$ would be a velocity. Any velocity profile as a function of time would serve to specify R equally well.

If the parameter is time, Eq. (1) or (2) becomes

$$R = \frac{\left((\mathrm{d}x/\mathrm{d}t)^2 + (\mathrm{d}y/\mathrm{d}t)^2\right)^{3/2}}{\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}} = \frac{V^3}{\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}} \tag{3}$$

where V is a freely chosen velocity at the point along the curve where the derivatives are taken. Equation 2 is true for any value of V. If V is chosen to be 1.0 in some arbitrary unit, then the denominator of the expression becomes $1/R$, which is the definition of "curvature" at the point. Any value of R is compatible with any value of V. There are at least two ways to illustrate that fact: a dimensional analysis, and the substitution of a few different velocity values into the equation.[1]

Dimensional analysis is a technique taught early in engineering school for checking the plausibility of an equation. For an equation to be plausible, the dimension of the expressions on the two sides of the equals sign or of an addition or subtraction sign must be the same. The symbols in the equation are described in terms of the basic units of physics, such as length (L), time (T), voltage (V), and mass (M), ignoring the units of measure.

In the case of (3) we need only L and T. We can substitute the dimensions for the components of (3) as follows: velocity $\rightarrow L/T$, acceleration $\rightarrow L/T^2$, yielding

$$L = \frac{\left((L/T)^2 + (L/T)^2\right)^{3/2}}{\frac{L}{T}\frac{L}{T^2} - \frac{L}{T}\frac{L}{T^2}} = \frac{(L/T)^3}{\left(L^2/T^3\right)} = L \tag{3a}$$

which is independent of the units of time and, therefore, of velocity. Equation (3), therefore, is true for all velocities if it is true for any (which Eq. (3) shows that it is).

[1] I thank Dr. Bruce Abbott for these suggestions (Personal Communication 2017.11.11).

The actual parametric equation for a curve is irrelevant. It matters for determining the actual radius of curvature, because it determines $\mathrm{d}x/\mathrm{d}s$ and the other derivatives with respect to s. But we ask next only what happens when V is changed in Eq. (3). Suppose V is doubled. The numerator of Eq. 3 is multiplied by 8, the velocities in x and y are doubled, and the accelerations in x and y are quadrupled, so the denominator is also multiplied by 8, leaving R unchanged. Again, any value of V is compatible with any value of R.

Any way the equation is examined, R and V are mathematically completely independent of each other, even if experiments suggest that in many situations they are not factually independent. The research question is why mathematical independence does not imply measured independence in those experimental and observational situations.

Marken and Shaffer's paper

Marken and Shaffer (2017) relied on Gribble and Ostry (1996) as their starting point for their analysis. Gribble and Ostry had used well-known formulae to determine the velocities and radii of curvature in their studies. They measured the actual velocity of movement by recording x, y, and t values, and reported this using the Newton's "dot" notation to represent time derivatives:

$$V = \sqrt{\dot{x}^2 + \dot{y}^2} = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2} \tag{4}$$

They then used Eq. (3) reported in the same dot notation, to determine the radius of curvature:

$$R = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \tag{5}$$

This is where Marken and Shaffer's critical mistake occurs. Using the logical fallacy of taking one instance to be the only possible instance of a set, they noted the visual similarity between the expression under the square root sign in (4) and the expression inside the bracket of (5) and treated them as being the same thing. In (4), however, $\dot{x}$($\mathrm{d}x/\mathrm{d}t$) and $\dot{y}$ ($\mathrm{d}y/\mathrm{d}t$) are values observed in an experiment and are used to compute the corresponding velocity, whereas in (5) they are arbitrary parameters, corresponding to any velocity whatever (including the observed velocity). Nevertheless, Marken and Shaffer assert that only the the velocity observed in the experiment can correspond to the $\dot{x}$ and $\dot{y}$ in the expression for R.

Had the two identical expressions, that under the square root in (4) and that inside the bracket of the numerator of (5), represented only the measured velocities squared, the rest of the Marken and Shaffer paper would have followed. By falsely claiming that the measured velocity is the only velocity that can be inserted in (5) to produce a correct value for R, Marken and Shaffer "discover" instead that velocity is mathematically a function of a simple length.

According to them, velocity is what it was measured to be because only that velocity is compatible with R in Eq. (5). They call the denominator of 5 "D" and refer to D as a cross-product correction factor. They then write their key Eq. (6), which is true for any value of V whatever, but which they claim to be true only for the value of V found in the experiment:

$$V = R^{1/3}D^{1/3} \tag{6}$$

Accordingly, they assert that measured values of the power law that depart from 1/3 are in error because they omit consideration of D.

But what actually is D? Here is a derivation of D in terms of "s", the distance along the curve to the point where the derivatives are measured. Recall that $x = x(s)$, $y = y(s)$ is the parametric description of the curve in purely spatial variables in Cartesian coordinates.

$$D = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\left(\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}^2x}{\mathrm{d}s^2}\right) = V^3\left(\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}^2x}{\mathrm{d}s^2}\right) \tag{7}$$

Equation (7) shows that D is V^3 times an expression in purely spatial variables. Marken and Shaffer's key Eq. (6), therefore, can be written $V = R^{1/3}\cdot V$ (function of spatial variables) or $V = R^{1/3}VS$. S turns out to be $(1/R)^{1/3}$ or $C^{1/3}$ where C is the usual definition of curvature, as can be seen by setting $V = 1.0$ in (3). Equation (6), therefore, resolves to $V = V$, which is true for all V.

Implications

In this section, I consider how Eq. (6) would affect the interpretation of the experimental results and the requirement for continuing research had Marken and Shaffer been correct, and had D actually been independent of V.

First, had Marken and Shaffer been correct, their finding would not have addressed the issue that has been the object of so much research: Why does the observed velocity of movement of a living organism along a curve so often approximate a power function of the local radius of curvature, and under what conditions does the power vary over such a wide range? These observations are made without reference to D, and need to be explained no matter how (or whether) D affects the "true" power.

Accepting the use of the "cross product correction factor" to produce the exact 1/3 power law would then translate the research question into a question of when and why D takes on the values it does. Marken and Shaffer claimed to have solved the 1/3 power question, but did not address the question of the variations in the measured power, beyond attributing the deviations from 1/3 to variations in D, which causes the "aberration" from the truth. The question of why and by how much D varies in specific situations would still have been left open. It would have been the old question cast in different words.

The "aberrations"—deviations from 1/3—can be quite large even when the power law is observed in an experiment. For example, Zago et al. (2017) quote Huh and Sejnowski (2015) as reporting a range of powers from about 0.1 to 0.66 for curves of different complexity, and replicated that part of the Huh and Sejnowski study with similar results (Fig. 2c in Zago et al. shows the power that relates angular velocity to curvature, which is one minus the power that relates the tangential velocity to radius of curvature). Zago et al. also show dramatic failures of the power law for some very simple analytic examples of curve-tracing.

Since the Marken and Shaffer analysis is actually not correct, the central research question about the value of the observed power might be why it has so often been near 1/3, rather than why it varies so widely. If a rationale is accepted for there being a power law relationship and for that power often being 1/3, then deviations from 1/3 need to be explained. The alternative remains as it has long been, a general problem about why a power law is ever found and why the power is what it is when a power law is observed.

Omitted variable bias

Returning to the Marken-Shaffer's paper, the authors next ask whether a statistical analysis based on the scattergram that produces the power law (a linear function in log–log space) will find that by including the "omitted variable"

D the "corrected" regression in log–log space will have a slope of 1/3. In this case, D, R, and random noise[2] (subject and measurement variability) are the only variables in question. It is always true that if one includes all the sources of variation in an analysis one will obtain a precise result, and that is what Marken and Shaffer find by incorporating the covariance between D and R in addition to the observed effect of R.

Let us examine what they actually did. First, they say that their key equation $V=R^{1/3}D^{1/3}$ implies that the effect of D is an independent omitted variable when considering the relationship between V and R. Next they use a statistical technique that uses the normalized covariance between the included variable (log R) and the omitted variable (log D) to "correct" the power relation between V and R by adding a value ∂ found using the statistical analysis. In log–log space the power relation is $\log(V)=k\log(R)$ where k is the observed power, but when the omitted variable is included, it is $\log(V) = (k+\partial)\log(R)$.

However, from the derivation of D in (3), $D=V^3/R$, so

$$1/3\log(D) = \log(V) - 1/3\log(R) \tag{8}$$

The whole equation then becomes

$$\log(V) = (k+\partial)\log(R) + \log(V) - 1/3\log(R), \tag{9}$$

and the correction ∂ is $1/3 - k$ apart from a negligible contribution from statistical variability (noise). In words, the "cross-product" correction is exactly enough to remove the observed effect of R on V, leaving only the tautology $\log(V)=\log(V)$. Marken and Shaffer have found the "omitted variable" for predicting V, and it is V, which is entirely contributed by their "cross-product" correction factor D, apart from the experimentally observed effect of R, for which their statistical analysis exactly compensates.

To prove their thesis, having found ∂ by statistical analysis, Marken and Shaffer simply leave the equation in its original form as $\log(V) = (k+\partial)\log(R) + 1/3\log(D)$, where $(k + \partial)$ was found statistically to be 1/3 in every case they presented. Wrongly asserting that D is independent of V, they assign the power relation to R, instead of to D where it belongs. Doing so, they always find the predicted result of exactly 1/3 for the "true" power of the power function relating V to R. This precision of slope—always exactly 1/3—should by itself have been a warning signal that something was wrong with their analysis. Experimentally measured relationships do not normally have such precision.

[2] Noise almost certainly has a covariance very near zero with any prespecified variable (such as D and R) so its contribution to the calculated slope is negligible.

Toy helicopter chase

Marken and Shaffer next propose a Perceptual Control Theory model (PCT: Powers 1973/2005, though they do not name the theory) to explain the power law that is observed when someone chases and finally catches a toy helicopter ($V=cR^{0.22}$).

In my personal opinion, Perceptual Control Theory is a powerful foundation for psychology, and ought to be able to explain the power law and the contextual variation in the observed power, though to date it has not. In this case, however, Marken and Shaffer misuse it. The model they offer may or may not be a correct PCT model for what people do when chasing toy helicopters, but either way, it contains and implies nothing that would explain why the movements of either people or toy helicopters conform to the power law. It simply asks how people act to bring their perceptions of the helicopter's position in x and y relative to their own position nearer to their reference values for those perceptions (equality).

If a power law relation exists between the velocity of the helicopter or the pursuer and the radius of curvature at points along their respective paths (and according to their data it does), the reason for that relationship is not addressed by their model, which appears to have been introduced only to bring the ideas (if not the name) of Perceptual Control Theory to the notice of a wider public (in itself a laudable objective, or so I believe). It is quite possible, even likely, that something about the processes involved in the control of certain perceptions accounts for the power law and the variations in the observed power, but Marken and Shaffer do not pursue this line of enquiry.

Final comment

This comment has two main purposes. First, to show that the analysis offered by Marken and Shaffer (2017) is logically flawed, and second to argue that had it been correct, it would have had no effect on the research questions surrounding the power law, namely when and why the power law is observed, and why the power is what it is when the power law is observed. Marken and Shaffer seem to want to prove that there is nothing to be studied, because the power-law itself is a "Behavioural Illusion". Whether it is or not, and whether their analysis is correct or not, the research questions would have been unaffected.

In PCT, "Behavioral Illusion" is a technical term, and that is how the authors use it. It implies that an observer or experimenter has interpreted the form of an observed effect to be a consequence of processing within the subject, whereas because of control the form of the effect is determined by properties of the subject's environment. The illusion is formed in the mind of the person who makes the interpretation.

"Side effect" is another common term used technically in PCT, where it has much the same meaning as it does in everyday language. A side effect is an observable effect that is not intended by the performer, but is a consequence of the performer acting on the environment to achieve something else entirely. The power law is almost certainly a side effect in any of the experiments that find velocity to have a near power law relationship with the radius of curvature, since it is very unlikely that any human, let alone a fly larva, acts with the intention of producing a power law relationship between travel speed and local curvature. Perhaps, it also creates a behavioural illusion in the minds of some theorist. Marken and Shaffer's paper sheds no light on that issue.

All in all, the initial simple mistake of taking a visual similarity to be a mathematical identity completely invalidates the rest of Marken and Shaffer's paper. The paper, therefore, contributes nothing but confusion to the research on the power law relationship between tangential velocity and local radius of curvature.

Compliance with Ethical Standards

Conflict of interest No financial interests exist in this area. Marken and the author have, however, belonged to the same mailing list on Perceptual Control Theory for a quarter-century, where the Marken–Shaffer theory of the power law was first presented for discussion. They have often disagreed during that time, including on the topic addressed by this comment.

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