Comments on Marken and Shaffer: The power law of movement: An example of a behavioral illusion.

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Many researchers who have studied movements, by different organisms, human and non-human, along curved paths under a variety of conditions, have found a consistent form of relation between the tangential (along-track) velocity V and the local radius of curvature R (see Zago et al. 2017 for many examples). The consistent relation is that $V \approx cR^k$, where k is a constant less than unity, often near 0.33, and c is a proportionality constant appropriate to the organism and the situation.

Marken and Shaffer claim to have found a mathematical argument that proves the true exponent always to be ⅓. Deviations from this value occur because researchers have omitted a critical correction "cross-product" factor that the authors label "D". This note questions the analysis offered by Marken and Shaffer.

Mathematical Background

On the face of it, it would be remarkable, even magical to derive a velocity simply from the local spatial shape of a curve that happened to be traced by a moving entity. Have Marken and Shaffer managed to perform the magic, or have they erred in their analysis? I argue that their "magical" derivation of velocities directly from shapes is stage magic, a mathematical sleight-of-hand caused, as in the usual magician's trick, by seeing one thing as something else entirely.

Here is the background for the trick. If a curve on a plane surface is described in x-y coordinates, the most straightforward way to describe it is to select a point on the curve and report the x and y values of the point at distance "s" along the curve from that arbitrary point. The values of x and y are separate functions of distance along the curve, linked only by the actual shape of the curve. These two functions, x = x(s) and y = y(s), provide what is called a "parametric representation" of the curve with s as the parameter. The local (signed) radius of curvature at a point on the curve depends on the first and second derivatives of x and y with respect to s, taken at that point:

$$R = \frac{((dx/ds)^2 + (dy/ds)^2)^{3/2}}{\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}} \quad \ldots(1)$$

It does not matter what continuous parameter, say "z", is used to specify the distance along the curve, so long as any value of z can be converted unambiguously into a specific point on the curve. If s = f(z) has that property, then z can be substituted directly for s in equation (1). To see this, note that dx/ds•ds/dz = dx/dz, and similarly for the second derivatives $(d^2y/ds^2)\bullet(ds/dz)^2 = d^2y/dz^2$.

$$R = \frac{((dx/ds)^2 + (dy/ds)^2)^{3/2}}{\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}} = \frac{((\frac{dx}{dz}\frac{dz}{ds})^2 + (\frac{dy}{dz}\frac{dz}{ds})^2)^{3/2}}{(\frac{dz}{ds})^3(\frac{dx}{dz}\frac{d^2y}{dz^2} - \frac{dy}{dz}\frac{d^2x}{dz^2})} = \frac{((dx/dz)^2 + (dy/dz)^2)^{3/2}}{\frac{dx}{dz}\frac{d^2y}{dz^2} - \frac{dy}{dz}\frac{d^2x}{dz^2}} \quad \ldots(2)$$

Of course, z could be any variable, including t (time) or any function of time with the same property of resolving to a single-valued function of s. If the parameter is time, equation (1) becomes

$$R = \frac{((dx/dt)^2 + (dy/dt)^2)^{3/2}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}} = \frac{V^3}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}} \quad \ldots(3)$$

where V is velocity. Equation 2 is true for any value of V, since V derives from an arbitrary relationship between s and t. Any value of R is compatible with any value of V.

Marken and Shaffer Paper

Marken and Shaffer used Gribble and Ostry (1996) as their guide. Gribble and Ostry had measured the actual velocity of movement by recording x, y, and t values, and reported this correctly, using the Newton's "dot" notation to represent time derivatives:

$$V = \sqrt{\dot{x}^2 + \dot{y}^2} = (\dot{x}^2 + \dot{y}^2)^{1/2} \quad \ldots(4)$$

They used the same dot notation, again correctly. to determine the radius of curvature:

$$R = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{(\dot{x}\ddot{y} - \dot{y}\ddot{x})} \quad \ldots(5)$$

Here's where the "magical" sleight-of-hand comes in. Marken and Shaffer noted the visual similarity between the representation of measured velocity in (4) and the parametric numerator of (5) and treated them as being the same thing. They asserted that the arbitrary velocity parameter that is cubed in the numerator of the expression for R (5) to be of necessity the measured velocity in an experiment, reported as in (4). By substituting the measured velocity, the expression for R remains numerically correct. They do their magic trick of linking velocity to a purely spatial property by asserting that the measured velocity is the only velocity for which R is correct, whereas it is actually correct for any velocity at all.

Following their argument a little further, Marken and Shaffer now define the denominator of (5) to be "D", and write their equation 5,

$$V=R^{1/3}D^{1/3} \quad \ldots(6)$$

Here is the derivation of D.

$$\begin{aligned} D &= \frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} \\ &= \left(\frac{ds}{dt}\right)^3\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right) \\ &= V^3\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right) \end{aligned} \quad \ldots(7)$$

Equation (6) is correct for any V, but they continue as though the measured V was the only value of V for which it is correct. They then assert that the reported values of the power law that depart from ⅓ are in error because they omit consideration of D, which they call the "cross-product" variable. Equation (7) shows that D is V^3 times an expression in purely spatial variables. Marken and Shaffer's equation (6) therefore becomes $V = V•R^{1/3}•$(*function of spatial variables*).

Implications

Before examining the implications that Marken and Shaffer draw from their assertion that the measured V is the only V that can make true equation (2), consider how equation (6) would affect the interpretation of the experimental results if D had actually been independent of V.

Firstly, had Marken and Shaffer been correct, it would not have affected the issue that has been the object of much research: *Why does the velocity of movement of a living organism along a curve so often approximate a power function of the local radius of curvature, and under what conditions does the power vary over such a wide range*? These observations are made without reference to D, and need to be explained no matter how D affects the "true" power. For example, Zago et al. (2017) quote Huh and Sejnowski (2015) as reporting a range from about 0.1 to 0.66 for curves of different complexity levels, and replicated that part of the study (their Figure 2c shows the power that relates angular velocity to curvature, which is one minus the power that relates the tangential velocity to radius of curvature).

Since the Marken and Shaffer analysis is not correct, the research question about the value of the observed power might well be why it has so often been near ⅓, rather than why it varies so widely. If a rationale is accepted for there being a power-law relationship and for that power normally being ⅓, then deviations from this value are aberrations that need to be explained. The alternative is a general problem about why a power-law occurs at all and why the power is what it is in any particular situation. Marken and Shaffer claimed to have solved the ⅓ power question, but did not address the "aberration" question other than to attribute the deviations to variations in $D = V^3R$. The question of why and by how much D varies in specific situations would still have been left open.

Omitted Variable Bias

Continuing with the Marken-Shaffer paper, they next ask about whether a statistical analysis based on the regression that produces the power law (a linear function in log-log space) will find that by including D the "corrected" regression in log-log space will have a slope of ⅓. It is always true that if one includes all the sources of variation in an analysis one will obtain a precise result, and that is what Marken and Shaffer find by incorporating the cross-correlation between D and R in addition to the observed effect of R.

Their own equation reproduced as (6) shows that $D = V^3/R$. In log-log space, $\log(D) = 3\log(V) - \log(R)$, so the cross correlation is -1, and the effects they are analyzing depend only on the intercept, $3\log(V)$. R has no mathematical relationship to V, which means they are predicting V from a knowledge of V, using the observed relationship between V and R to eliminate its influence from the analysis. Hence when they include D in the analysis along with R, they always find the predicted result of exactly 1/3 for the "true" power of the power function, leaving only 7% of the variance to be accounted for by noise effects due to measurement error and inconsistency of the mover. This precision of slope — always exactly ⅓ — in itself should have been a warning signal that something was wrong with their analysis.

Toy helicopter chase

Marken and Shaffer conclude by proposing what they call a Perceptual Control Theory model to explain the power law that is observed when someone chases and finally catches a toy helicopter.

In my personal opinion, Perceptual Control Theory (PCT: Powers 1973, 2005) is a powerful foundation for psychology, and ought to be able to explain the power law and the contextual variation in the observed power. In this case, however, Marken and Shaffer misuse it. The model they offer may or may not be a correct PCT model for what people do when chasing toy helicopters, but whether it is or not, it contains nothing that would link the movements of either people or toy helicopters to the power law. It simply asks how people act in order to bring their perceptions of the helicopter's position in x and y to their reference values for those perceptions.

If a power-law relation exists between the velocity of the helicopter or the pursuer and the radius of curvature of either path (and according to their data it does), that relationship not addressed by their model, which appears to have been introduced only to bring the name of Perceptual Control Theory to the notice of a wider public (in itself a laudable objective, or so I believe).

Final Comment

The title of Marken and Shaffer's paper says that the Power Law is a "Behavioral illusion". In PCT "Behavioral Illusion" is a technical term that implies that an observer or experimenter has interpreted the form of an observed effect to be a consequence of processing within the subject, whereas because of control the form of the effect is determined by properties of the subject's environment. The power law may or may not be a behavioral illusion. Marken and Shaffer's paper sheds no light on that issue.

A common term used technically in PCT is "side-effect", which has much the same meaning as in everyday language. A side effect is an observable effect that is not intended by the performer, but is a consequence of the performer acting on the environment while intent on achieving something else. The power law is almost certainly a side-effect, since it is very unlikely that any human, let alone a fly larva, acts in order to produce a power law relationship between travel speed and local curvature.

All in all, the initial simple mistake of taking a visual similarity to be a mathematical identity completely invalidates the rest of Marken and Shaffer's paper. The paper therefore contributes nothing but confusion to the research on the power law relationship between tangential velocity and local radius of curvature.

References

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