Example

Disturbance is a sinusoid: $\left(d = \sin(wt + \phi)\right)$

Then for a linear system, perception must be a sinusoid of the same frequency $p = m\sin(wt)$

$\frac{1}{m}$ is the "control ratio", $C.R = \frac{|d|}{|p|}$

Using the standard method of analyzing the control loop:

$$p = d + o, = d + k\int e\,dt = d + k\int (r - p)\,dt = d - k\int p\,dt$$

$$d = p + k\int p\,dt$$

$$\sin(wt + \phi) = m\sin(wt) - \frac{km}{w}\cos(wt)$$

$$\frac{1}{m}\sin(wt + \phi) = \sin(wt) - \frac{k}{w}\cos(wt)$$

Set $c = \frac{-k}{w}$ and $x = wt$

$$\frac{1}{m}\sin(x + \phi) = \sin(x) + c\cos(x)$$

From the generic formula:

$A\cos x + B\sin x = C\sin(x + \theta)$, where $C = \left(A^2 + B^2\right)^{\frac{1}{2}}$ and $\theta = \arctan\left(\frac{B}{A}\right) = \arcsin\left[\frac{B}{\sqrt{A^2 + B^2}}\right]$

we have:

$$\phi = \arcsin\left[\frac{1}{\sqrt{1 + c^2}}\right]$$

The "control ratio" $C.R. = \frac{1}{m} = \sqrt{1 + c^2} = \frac{\sqrt{w^2 + k^2}}{w}$

At high frequencies, the control ratio $\frac{1}{m} \to 1$ (no control), and at low frequencies $\frac{1}{m} \to \frac{k}{w}$

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Now consider the correlation between p and d, which we obtain by integrating the cross product of the scaled sinusoids for p and d.

Scaling the components: $\int_0^{2\pi} \sin^2(x)\,dx = \pi$, so scale p by $\sqrt{\pi}$ and d by $m\sqrt{\pi}$ to bring both to unity scale

$$\text{correl (p.d)} = \frac{1}{\pi}\int_0^{2\pi} \sin(x + \phi)\cos(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} (\sin(2x + \phi) + \sin(\phi))\,dx$$

$$= 0 + \frac{1}{2\pi\sqrt{1 + c^2}}\int_0^{2\pi} dx = \frac{1}{\sqrt{1 + c^2}}$$

The correlation is $\frac{1}{C.R}$