Comments on Marken and Shaffer: The power law of movement: An example of a behavioral illusion.

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Abstract

Many researchers who have studied movements along curved paths, under a variety of conditions, by different organisms, human and non-human, have found a consistent form of relation between the tangential (along-track) velocity V and the local radius of curvature R (see Zago et al. 2017 for many examples). The consistent relation is that $V \approx cR^k$, where k is a constant less than unity, often near 0.33, and c is a proportionality constant appropriate to the organism and the situation.

Marken and Shaffer claim to have found a mathematical argument that proves the true exponent of the power relating velocity and radius of curvature always to be ⅓. They say that deviations from this value occur because researchers have omitted a critical correction "cross-product" factor that the authors label "D". This note questions the analysis offered by Marken and Shaffer.

Mathematical Background

On the face of it, it would be remarkable, even impossible, to derive a velocity directly from a length measure (the radius of curvature). Have Marken and Shaffer managed to do the impossible, or have they erred in their analysis? I argue that their derivation of velocities directly from shapes is faulty, an error produced by treating one instance of a large set as being the only member of the set.

Here is the mathematical background for the mistake. A curve on a plane surface can be described by selecting an arbitrary point on the curve and reporting the x and y values of other points as functions of distance "s" along the curve from that arbitrary point. The values of x and y are independent functions of distance along the curve, linked only by the actual shape of the curve. These two functions, $x = x(s)$ and $y = y(s)$, produce what is called a "parametric representation" of the curve with s as the parameter. The local (signed) radius of curvature at any point on the curve depends on the first and second derivatives of x and y with respect to s at that point according to a long-known expression:

$$R = \frac{((dx/ds)^2 + (dy/ds)^2)^{3/2}}{\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}} \quad \ldots(1)$$

The parameter need not be the along-curve distance "s" itself. Any arbitrary continuous variable, say "z", may be used to specify the distance along the curve, if its value can be converted unambiguously into a specific value of s. If s = f(z) has that property, then z can be substituted directly for s in equation (1).

To see this, note that (dx/ds)(ds/dz) = dx/dz, while for the second derivatives $(d^2y/ds^2)(ds/dz)^2 = d^2y/dz^2$. Hence,

$$\begin{aligned} R &= \frac{((dx/ds)^2 + (dy/ds)^2)^{3/2}}{\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}} \\ &= \frac{((\frac{dx}{dz}\frac{dz}{ds})^2 + (\frac{dy}{dz}\frac{dz}{ds})^2)^{3/2}}{(\frac{dz}{ds})^3(\frac{dx}{dz}\frac{d^2y}{dz^2} - \frac{dy}{dz}\frac{d^2x}{dz^2})} \\ &= \frac{((dx/dz)^2 + (dy/dz)^2)^{3/2}}{\frac{dx}{dz}\frac{d^2y}{dz^2} - \frac{dy}{dz}\frac{d^2x}{dz^2}} \end{aligned} \quad \ldots(2)$$

which is just (1) with "z" replacing "s". Of course, z could be any variable, including t (time) before or after some arbitrary starting moment at which s is defined to be zero, in which case ds/dt would be a velocity. Any velocity profile as a function of time would serve the purpose. If the parameter is time, equation (1) becomes

$$R = \frac{((dx/dt)^2 + (dy/dt)^2)^{3/2}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}} = \frac{V^3}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}} \quad \ldots(3)$$

where V is the chosen velocity at the moment and position where the derivatives are taken. Equation 2 is true for any value of V, since V was arbitrarily chosen. Any value of R is compatible with any value of V.

There are at least two ways to illustrate that this is true, a dimensional analysis, and the substitution of a few different velocity values into the equation for a given radius[1].

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1. I thank Dr. Bruce Abbott for these suggestions (Personal Communication 2017.11.11).

Dimensional analysis is a technique taught early in engineering school for checking the plausibility of an equation. The symbols in the equation are described in terms of the basic units of physics, such as length (L), time (T), voltage (V), mass (M), etc., ignoring the units of measure. In the case of (3) we need only L and T. We can substitute the dimensions for the components of (3) as follows: velocity → L/T, acceleration → L/T², yielding

$$L = \frac{((L/T)^2 + (L/T)^2)^{3/2}}{\frac{L}{T}\frac{L}{T^2} - \frac{L}{T}\frac{L}{T^2}} = \frac{(\frac{L}{T})^3}{\frac{L}{T}\frac{L}{T^2} - \frac{L}{T}\frac{L}{T^2}} = L \quad \text{...(3a)}$$

which is independent of the units of time, and therefore of velocity. Equation (3) therefore is true for all velocities if it is true for any (which equation (3) shows that it is).

The actual parametric equation for a curve is irrelevant. It matters for determining the actual radius of curvature, because it determines dx/ds and the other derivatives with respect to s. But we ask only what happens when V is changed in equation (3). Suppose V is doubled. Then the numerator of equation 3 is multiplied by 8, the velocities in x and y are doubled, and the accelerations in x and y are quadrupled, so the denominator is also multiplied by 8, leaving R unchanged. Again, any value of V is compatible with any value of R.

Any way the equation is examined, R and V are mathematically completely independent of each other, even if experiments suggest they are not factually independent in many situations. The research question is why mathematical independence does not imply measured independence in those experimental and observational situations.

Marken and Shaffer Paper

Marken and Shaffer used Gribble and Ostry (1996) as their starting point for their analysis. Gribble and Ostry had measured the actual velocity of movement by recording x, y, and t values, and reported this correctly, using the Newton's "dot" notation to represent time derivatives:

$$V = \sqrt{\dot{x}^2 + \dot{y}^2} = (\dot{x}^2 + \dot{y}^2)^{1/2} \quad \text{...(4)}$$

They then used the same dot notation, again correctly, as shown in equation (3), to determine the radius of curvature:

$$R = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{(\dot{x}\ddot{y} - \dot{y}\ddot{x})} \quad \text{...(5)}$$

This is where Marken and Shaffer's critical mistake occurs. They noted the visual similarity between the expression under the square root sign in (4) and the expression inside the bracket of (5) and treated them as being the same thing. In (4) $\dot{x}$ and $\dot{y}$ are determined by the velocity observed in an experiment, whereas in (5) they are arbitrary parameters, valid for any velocity whatever (including the observed velocity), which is not the same at all. Had the two expressions both represented only the measured velocities, the rest of the Marken and Shaffer paper would have followed, but they do not. That fact invalidates the rest of the paper, so it is necessary to examine the issue closely.

When the measured time is used to generate the velocity and acceleration derivatives in the expression for R, the calculation of R remains correct, since it is true for all velocities. Marken and Shaffer assert instead that velocity is necessarily a particular function of a length, by falsely claiming that the measured velocity is the only velocity that can be inserted in (5) that would produce a correct value for R.

Following their argument a little further, Marken and Shaffer next define a variable “D” as the denominator of (5). They call D a cross-product correction factor, and write their key equation,

$$V=R^{1/3}D^{1/3} \quad \ldots(6)$$

They then assert that measured values of the power law that depart from ⅓ are in error because they omit consideration of D. But what actually *is* D? Here is the derivation of D from the fundamental parametric description of the curve in terms of the x-y values as functions of distance along it.

$$\begin{aligned} D &= \frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} \\ &= \left(\frac{ds}{dt}\right)^3\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right) \\ &= V^3\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right) \end{aligned} \quad \ldots(7)$$

Equation (7) shows that D is V^3 times an expression in purely spatial variables. Marken and Shaffer’s key equation (6) therefore can be written $V = R^{1/3}$•V•(*function of spatial variables*), and the “function of spatial variables” is $S = (1/R)^{1/3}$ or $C^{1/3}$ where C is the usual definition of curvature.

Implications

Before examining the implications that Marken and Shaffer draw from their assertion that the measured V is the only V for which equation (2) is valid, consider how equation (6) would affect the interpretation of the experimental results and the requirement for continuing research had they been correct, and had D actually been independent of V.

Firstly, had Marken and Shaffer been correct, it would not have affected the issue that has been the object of much research: *Why does the observed velocity of movement of a living organism along a curve so often approximate a power function of the local radius of curvature, and under what conditions does the power vary over such a wide range*? These observations are made without reference to D, and need to be explained no matter how (or whether) D affects the "true" power. Accepting the use of the "cross product correction factor" to produce the ⅓ power law exactly would then translate the research question into a question of when and why D takes on the values it does. Marken and Shaffer claimed to have solved the ⅓ power question, but did not address the question of the variations in the measured power other than to attribute the deviations from ⅓ to variations in D. The question of why and by how much D varies in specific situations would still have been left open.

The "aberrations" can be quite large. For example, Zago et al. (2017) quote Huh and Sejnowski (2015) as reporting a range of powers from about 0.1 to 0.66 for curves of different complexity levels, and replicated that part of the Huh and Sejnowski study with similar results (Figure 2c in Zago et al. actually shows the power that relates angular velocity to curvature, which is one minus the power that relates the tangential velocity to radius of curvature).

Since the Marken and Shaffer analysis is actually not correct, the research question about the value of the observed power might rather be why it has so often been near ⅓, rather than why it varies so widely. If a rationale is accepted for there being a power-law relationship and for that power normally being ⅓, then deviations from ⅓ need to be explained. The alternative is (as it has been) a general problem about why a power-law occurs at all and why the power is what it is in any particular situation.

Omitted Variable Bias

Continuing with the Marken-Shaffer paper, they next ask about whether a statistical analysis based on the regression that produces the power law (a linear function in log-log space) will find that by including the "omitted variable" D the "corrected" regression in log-log space will have a slope of ⅓. In this case, D, R, and random noise (subject and measurement variability) are the only variables in question. It is always true that if one includes all the sources of variation in an analysis one will obtain a precise result, and that is what Marken and Shaffer find by incorporating the cross-covariance between D and R in addition to the observed effect of R.

Let us examine what they actually did. Firstly they say that their key equation $V = R^{1/3}D^{1/3}$, implies that the effect of D is an independent omitted variable when considering the relationship between V and R. Next they use a statistical technique that uses the normalized covariance between the included variable (log R) and the omitted variable (log D) to "correct" the power relation between V and R by adding a value ∂ found by the statistical analysis. In log-log space the power relation is $\log(V) = k \log(R)$ where k is the observed power, but when the omitted variable is included, it is $\log(V) = (k+\partial)\log(R)$.

However. we know from (7) that $D = V^3/R$, so $⅓ \log(D) = \log(V) - ⅓ \log(R)$. The whole equation then becomes $\log(V) = (k+\partial) \log(R) + \log(V) - ⅓ \log(R)$, and the correction ∂ is ⅓ - k apart from a trivial contribution from statistical variability (noise) that averages out both over the duration of a trial run and over the set of trial runs. In words, the "cross-product" correction completely removes the effect of the experimental observations except for the increasingly negligible noise, leaving only the tautology $\log(V) = \log(V)$.

Marken and Shaffer, however, having found ∂ by statistical analysis, leave the equation in its original form as $\log(V) = (k+\partial) \log(R) + ⅓ \log(D)$. By asserting that D is independent of V, they assign the power relation to R, instead of to D where it belongs. Doing so, they always find the predicted result of exactly ⅓ for the "true" power of the power function relating V to R. This precision of slope — always exactly ⅓ — should by itself have been a warning signal that something was wrong with their analysis. Experimentally measured relationships do not normally have such precision.

Toy helicopter chase

Marken and Shaffer conclude by proposing a Perceptual Control Theory model (PCT: Powers 1973, 2005, though they do not name the theory) to explain the power law that is observed when someone chases and finally catches a toy helicopter ($V = cR^{0.22}$).

In my personal opinion, Perceptual Control Theory is a powerful foundation for psychology, and ought to be able to explain the power law and the contextual variation in the observed power, though to date it has not. In this case, however, Marken and Shaffer misuse it. The model they offer may or may not be a correct PCT model for what people do when chasing toy helicopters, but whether it is or not, it contains nothing that would explain why the movements of either people or toy helicopters conform to the power law. It simply asks how people act in order to bring their perceptions of the helicopter's position in x and y relative to their own position nearer to their reference values for those perceptions (equality).

If a power-law relation exists between the velocity of the helicopter or the pursuer and the radius of curvature of either path (and according to their data it does), the reason for that relationship is not addressed by their model, which appears to have been introduced only to bring the ideas (if not the name) of Perceptual Control Theory to the notice of a wider public (in itself a laudable objective, or so I believe). It is quite possible, even likely, that something about the processes involved in the control of certain perceptions accounts for the power law and the variations in the observed power, but Marken and Shaffer do not pursue this line of enquiry.

Final Comment

The title of Marken and Shaffer's paper says that the Power Law is a "Behavioral illusion". In PCT, "Behavioral Illusion" is a technical term, and that is how the authors use it. It implies that an observer or experimenter has interpreted the form of an observed effect to be a consequence of processing within the subject, whereas because of control the form of the effect is determined by properties of the subject's environment. The illusion is in the mind of the theorist who makes the interpretation.

"Side effect" is another common term used technically in PCT, where it has much the same meaning as it does in everyday language. A side effect is an observable effect that is not intended by the performer, but is a consequence of the performer acting on the environment to achieve something else entirely. The power law is almost certainly a side-effect in any of the experiments that find velocity to have a near power-law relationship to the radius of curvature, since it is very unlikely that any human, let alone a fly larva, acts in order to produce a power law relationship between travel speed and local curvature. Perhaps it also creates a behavioural illusion. Marken and Shaffer's paper sheds no light on that issue.

All in all, the initial simple mistake of taking a visual similarity to be a mathematical identity completely invalidates the rest of Marken and Shaffer's paper. The paper therefore contributes nothing but confusion to the research on the power law relationship between tangential velocity and local radius of curvature.

References

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Conflict of Interest Statement

No financial interests exist in this area. Marken and the author have, however, belonged to the same mailing list on Perceptual Control Theory for a quarter-century, and have often disagreed during that time, including on the topic addressed by this comment, where the Marken-Shaffer theory of the power law was first presented for discussion.