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Relation Between Velocity and Curvature in Movement: Equivalence and Divergence Between a Power Law and a Minimum-Jerk Model

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Unconstrained hand movements typically display a decrease in hand speed around highly curved sections of a trajectory. It has been suggested that this relation between tangential velocity and radius of curvature conforms to a one-third power law. We demonstrate that a one-third power law can be explained by models taking account of trajectory costs such as a minimum-jerk model. Data were analyzed from 6 subjects performing elliptical drawing movements of varying eccentricities. Conformity to the one-third power law in the average was obtained but is shown to be artifactual. It is demonstrated that asymmetric velocity profiles may result in consistent departures from a one-third power law but that such differences may be masked by inappropriate analysis procedures. We introduce a modification to the original minimum-jerk model by replacing the assumption of a Newtonian point-mass with a visco-elastic body. Simulations with the modified model identify a basis for asymmetry of velocity profiles and thereby predict departures from a one-third law commensurate with the empirical findings.

Velocity and Curvature Effects

Hand trajectories in free space demonstrate remarkable reliability and are generated with apparent simplicity, given the computational complexities encountered. The apparent ease with which human actions are controlled belies the difficulty in programming an anthropomorphic robot for similar tasks in a changing environment. An understanding of the nature of such control and the generation of adequate descriptive models are both contingent upon empirical observations of stable features of such trajectories and upon their relevance to the efficacy of movement. In this respect, seemingly minor features of everyday movements may provide the key to evaluating the adequacy of more global descriptions of human control. The scaling of velocity with movement time and distance, for instance, has been well documented and is generally accepted as relevant to the modeling of human motor control. A rather more esoteric observation has been the precise relation between instantaneous velocity and local curvature in a trajectory. This latter feature has received less general attention but provides a local description of the unfolding of planned movements and as such provides a potentially powerful tool for the appraisal of motor models or emergent control.

The dependence of velocity upon trajectory curvature was noted in a number of early studies of human upper-limb control (Binet & Courtier, 1893; Jack, 1895). Jack (1895) made the specific observation that "velocity of a curve varies, roughly speaking, with the radius of curvature" (p. 476). These early observations were given reemphasis through the work of Viviani and colleagues who proposed the "isogony principle" (Viviani & Terzuolo, 1982) and the "two-thirds power law" (Lacquaniti, Terzuolo, & Viviani, 1983). The isogony principle is based on the observation that within a "unit of action," equal angles are described in similar times (even though arc length may vary). The two-thirds power law can be seen as qualifying Jack's (1895) observation by proposing a specific relation between the geometric properties of the trajectory (curvature, C) and the kinematics (angular velocity, A) of the movement:

$$A(t) = K.C(t)^{2/3} \qquad |C| > 0. \tag{1}$$

This may also be written in terms of tangential velocity (V) and the radius of curvature (R) and empirically tested through regression of the natural logarithms of V and R:

$$V(t) = K.R(t)^{1/3} \qquad |R| < \infty \tag{2}$$

$$\log_e(V) = \log_e(K) + \frac{1}{3}\log_e(R). \tag{3}$$

This power law serves to describe a generally observable characteristic of upper limb movements: Hand speed drops when negotiating highly curved portions of a trajectory, in a

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manner similar to that of a vehicle negotiating a curved road. The extent to which speed (tangential velocity) is scaled down is described by the exponent in Equation 2. An increase in this exponent means greater scaling, whereas if the exponent approaches zero, velocity becomes independent of curvature and is purely dependent upon the global scaling of the velocity gain factor (K).

The relevance of such a mathematical description, when we consider the development of movement control, is somewhat vague. There has been little insight into how or why such a relation might emerge within an ecologically bounded system. One interesting observation by Lacquaniti et al. (1983) was that the one-third condition is satisfied if curved motion is produced by coupling two orthogonal (i.e., x, y) sinusoids, independent of phase (at 0° and 180° phase, motion is along a straight line and hence outside the limits of Equation 2).

At first impression, this seems an interesting observation. We feel it is prudent, however, to be critical with regard to the functional significance of sinusoidal motion. There is little in the way of evidence to suggest that the sinusoid has biological rather than mathematical relevance. This may seem a contentious statement, given the wealth of research that has utilized the principles of harmonic oscillators (e.g., Hollerbach, 1980, 1981; Kelso, Holt, Rubin, & Kugler, 1981). The underpinning of such work has been the parallel between the behavior of the human motor system and that of an idealized mechanical system, such as an undamped mass spring, that conforms to sinusoidal motion. A literal transposition from this mechanical model to human behavior, however, is obviously unjustified as Saltzman and Kelso (1987; note 4) pointed out. Hollerbach (1981) also emphasized that the sinusoid was adopted purely for "mathematical convenience" (pp. 6, 10), and his model is applicable to a wide range of oscillations. Sinusoidal motion has been used as a convenient and useful approximation of human oscillatory motion. We suggest, however, that there has been little to substantiate its physiological rather than mathematical relevance. It is perhaps also relevant to note that recent motor models using nonlinear oscillators such as the Van der Pol equation (see Kay, Kelso, Saltzman, & Schöner, 1987) also depart from such motion. Sinusoidal motion can be approximated only by a Van der Pol oscillator with a very small escapement component, which in turn undermines the advantages of a stable limit cycle in its response to external perturbations.

We suggest that it is more useful to acknowledge that the one-third law may hold for a wide range of coupled oscillations, many of which may be perceived as having greater psychological or biological logic underpinning them, than to propose that the motor system conforms to pure sinusoidal motion for the production of oscillatory movement. This statement does not undermine the validity of research analyzing motion in terms of coupled sinusoidal components (e.g., Soechting, Lacquaniti, & Terzuolo, 1986; Soechting & Terzuolo, 1986) but emphasizes that such an approach is essentially a description rather than a theory of control. Such a description based on a sine wave provides a yardstick for appraising deviations from regularity. Soechting et al. (1986) undertook such an approach and noted the deviations from sinusoidal motion of wrist velocity, ranging from 8% to 37% along a single axis, which obviously leaves scope for a wide range of alternative models.

Equivalence of Minimum-Jerk Motion to a Power Law

An interesting criterion for the organization of biological motion is the minimum-jerk model proposed by Flash and Hogan (1985). This model provides a closed-form solution to the problem of negotiating a given trajectory while minimizing transients in acceleration (jerk). For point-to-point arm movements along a curved trajectory, Flash and Hogan (1985) have shown that this model can exhibit behavior equivalent to the isochrony principle of Viviani and Terzuolo (1982). The attraction of such a model is that it can provide a suitable time history for an acceleration function (from which joint torques may be derived), based on a measure of kinematic smoothness. From introspection, smoothness of movement seems to be something that the individual is normally aware of during or following motion. Hence, we might conceive of a child refining trajectories by trial-and-error learning until finally arriving at those which feel smooth. Despite evidence that jerk costs may be reduced during development (Wann, 1987), this plausible argument has a major flaw. The criterion chosen by Flash and Hogan (1985) was the minimization of the mean-squared value of the third derivative of position. Primary sources of afference in the human body provide information about relative position (joint angle receptors) or stretch and rate of stretch (muscle spindles). The crucial question is how a system that primarily senses joint angle, stretch, and rate of stretch could know about third derivatives. In support of this approach, it can be demonstrated (Nimmo-Smith, 1988; Wann & Nimmo-Smith, 1988) that a model exploiting the visco-elastic nature of the human limb (related to ideas presented briefly later in this article) can provide perceptual information equivalent to that required by a developmental application of the Flash and Hogan model.

A less constrained version of the minimum-jerk model was described by Nelson (1983). This version allows acceleration to be non-zero at the movement endpoints and hence to satisfy the conditions normally present during repetitive, back-and-forth motion. The solution to the minimization is a quintic polynomial, whose coefficients are determined by the boundary conditions on $x(t)$ and its higher derivatives (Nelson, 1983). Figure 1 compares the effect of coupling two such quintic motions with two coupled sinusoidal motions (both at 90° phase) to produce an ellipse (eccentricity = .80, cyclic frequency 1 Hz). It is obvious that the two motions would be empirically undistinguishable, and both approaches result in a one-third power law for the velocity-radius of curvature relation. It should also be noted that other models including "minimum energy" (Nelson, 1983) and "best stiffness" (Hasan, 1986) may produce very similar results.

It is worth considering what conditions would be necessary within the human motor system to generate either motion. Sinusoidal motion is sustained within an undamped mass-spring system, but as already discussed, this can be considered

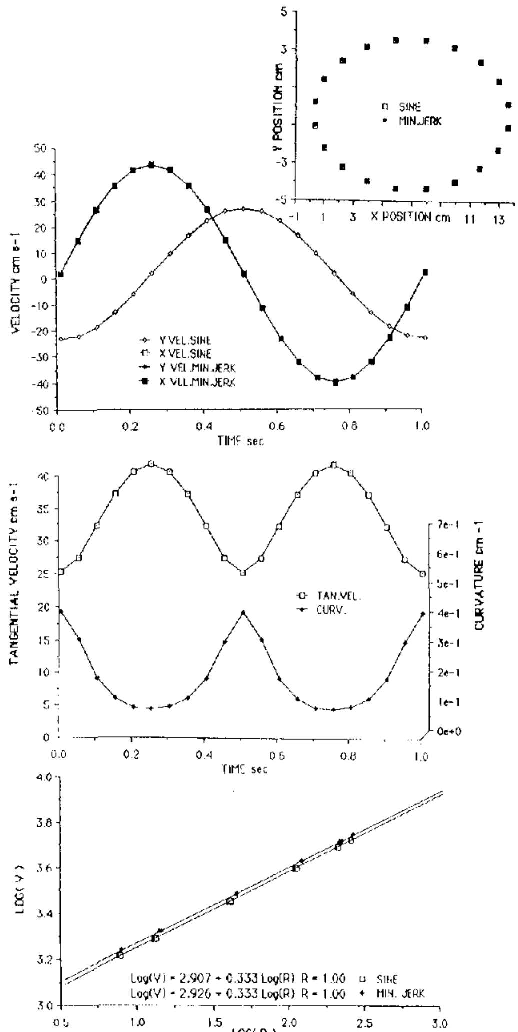

Figure 1. Simulated elliptical motion to compare a minimum-jerk model with sinusoidal motion. Inset: Spatial patterns for a .80 ellipse performed at 1-Hz cyclic frequency. (SINE = simulation with sinusoids; MINJERK = minimum-jerk model.) Upper panel: Overlaid velocity profiles for both models. (XVEL. = velocity along the x-axis; Y VEL. = velocity along the y-axis for both simulations.) Middle panel: Tangential velocity (TAN. VEL.) and curvature (CURV.) records for the minimum-jerk model, displaying the reciprocal relation typical of a power law. Lower panel: Regression of the natural logarithms of tangential velocity and radius of curvature for both models, to demonstrate the equivalence to a ⅓ power law. (Minimum-jerk model adapted from equation in Nelson, 1983).

only as a weak metaphor of human muscular control. A more acceptable and widely used model is one that acknowledges the presence of inherent dissipative forces, such as a linear second-order, damped mass-spring model. Cyclical motion may be maintained in such a system by reciprocal shifts in the equilibrium point at each half-cycle, but the damping term introduces skewness into the sinusoidal motion. A further option is the implementation of a nonlinear oscillator that can balance the energy losses within the system to maintain a stable cycle. Despite being an interesting theoretical suggestion, such limit cycles lack any physiological underpinnings with respect to human muscular control (Kay et al., 1987, p. 189). In addition, as discussed earlier, sinusoidal motion arises in limit cycle systems, such as the Van der Pol (Kay et al., 1987), only if they have low stability.

In contrast, the minimum-jerk approach is not contingent upon a specific model of the muscular system. In a simple damped mass-spring system, minimum-jerk trajectories may be generated by specifying a suitable virtual trajectory (time history of the shift of the equilibrium point). In principle, this may be through the reciprocal action of agonists and antagonists, while maintaining constant stiffness (Hogan, 1984), or through the modulation of both reciprocal drives and coactivation with time (Berkinblit, Feldman, & Fukson, 1986; Feldman, 1981). Cyclical motion may be maintained by a cyclic reiteration of the virtual trajectory, with trajectory corrections to perturbations resulting from the self-equilibrating properties of such a system, (Bizzi, Accornero, Chapple, & Hogan, 1984).

The simulations of elliptical trajectories in this article were produced by linking two minimum-jerk "oscillations" (piecewise polynomial periodic functions) at 90° phase. The four via-points of an ellipse (extremes of the x,y axes) can be achieved by scaling the amplitude for orthogonal motions (scaling the virtual trajectory) without scaling the natural frequency of the system. In this respect, the minimum-jerk model mimics the behavior of the sinusoidal model proposed by Lacquaniti et al. (1983). The primary difference between the two approaches, however, is that minimum-jerk is a prescription for movement, based upon assumptions of human ontogeny and intent, that exploits the properties of the muscular system, whereas a sinusoidal model is the simplest and most abstract (mathematical) description of motion but is lacking in physiological or psychological rationale.

Evidence of a One-Third Power Law

The one-third power law for elliptical motion is satisfied by coupled sinusoidal oscillations. The minimum-jerk simulation produces equivalent results to a sinusoidal model because the quartic and cubic velocity and acceleration functions closely approximate sine/cosine waves (Figure 1). We have argued that the sinusoid approach is an arbitrary model, in the sense that it appears to approximate the general form of human oscillatory motion but lacks clear rationale. If human oscillations during such tasks depart from the ideal of sinusoids, then we might anticipate that such a "law" will waver from its one-third value. Curved motion produced with triangular velocity profiles, for instance, can be shown to exhibit a velocity-curvature power exponent of a similar magnitude but to deviate from a one-third law. It is apparent, therefore, that if we are to accept the power law of Lacquaniti and colleagues as indicative of "the control logic" (1983, p. 130) and thereby supportive of models which predict it, any deviations from the law should be examined for evidence of a principled basis.

A second reason for the close examination of such a law is that previous articles in this *Journal* have proposed it both as a basis for the analysis of segmentation in movement (Viviani

& Cenzato, 1985) and as a feature for the appraisal of tracking behavior (Viviani, Campadelli, & Mounoud, 1987). In the former study, the one-third law was accepted a priori in order to calculate the velocity gain factor K (Equation 2).

A full reading of the evidence for the one-third law (two-thirds for Equation 1) is not totally convincing. Data published by Viviani and McCollum (1983) on the drawing of ellipses demonstrated a power function with a radius exponent closer to ¼ rather than ⅓ (p. 216), whereas the data presented by Lacquaniti et al. (1983) displayed curvature exponent values that included ⅔ (Equation 1) but ranged from 0.576 to 0.824 within a single subject, dependent upon speed (p. 122). In addition, when a ⅔ power function was fitted to the curvature data from a range of ellipses, little more than half the data points were accounted for, in that the data displayed marked nonlinearity for curvatures with a radius above 10 cm (p. 123). Given that the radius of curvature for pure wrist and pure elbow motion may be around 15 cm and 45 cm, respectively (measured at a pencil point held in a tripod grip), the aforementioned limit seems rather low. The findings of other researchers are mixed; for example, Thomassen and Teulings (1985) concluded that a one-third law may hold for simple drawing movements but is not robust for more complex patterns or for normal handwriting.

Given these inconsistencies, the validity of the one-third power law seems to be in doubt. The aim of this article is to examine deviations from the one-third power law—first, in terms of their significance to the stability of such a description and, second, to ascertain whether such deviations may provide insights into the control logic beyond those already discussed with respect to conformity to such a law. This latter aspect will encompass an expansion of the notion of minimum-jerk motion in human movement.

Method

General

There are few external restrictions upon arm trajectories in everyday behaviors. Many actions are targeted and hence have specified endpoints and distance, whereas some may have some general shape requirements (e.g., obstacle avoidance, handwriting, and drawing). Few have rigid time constraints, other than in terms of general tempo. Exceptions would be hitting or catching moving objects, both of which may be properly understood only in terms of externally driven timing rather than endogenous organization. With these factors in mind, the testing procedure was structured so that as few constraints as possible were placed upon the subjects' performance. Subjects were required to reproduce elliptical movements of different eccentricities (to provide different curvatures), but in contrast to previous studies (Lacquaniti et al., 1983; Viviani & Cenzato, 1985), subjects were not presented with a raised template or predrawn ellipse, and execution speed was not cued but left to the subjects' preference. The analyses centered therefore on the velocity–curvature relation during trajectories that were specified in general shape and size but for which local variations in shape and speed were permitted.

Subjects

Six volunteer subjects took part in the study (3 female, 3 male; age range: 20–40 years). Three were paid. All subjects were naive to the nature of the experiment and what was to be measured. None of the authors were involved as subjects.

Apparatus

Ellipses were drawn on a Micropad pressure-sensitive pad (Quest Micropad Ltd., Dorset, U.K.). This device allows writing movements that are performed with any type of object to be recorded at 200 Hz with a measured "combined planar accuracy" (Teulings & Maarse, 1984) of less than 0.2 mm. The writing implement was a normal 7-mm-diameter barrel, HB pencil with no auxiliary attachments.

Procedure

Four small circles were presented on a sheet of paper so that each circle marked the horizontal and vertical extremes of an imaginary ellipse, aligned with its greater semi-axis along the horizontal. The distance between marks was chosen so that an elliptic trajectory through all four points would yield patterns of similar perimeter (32 cm) with eccentricities of .60, .80, .90, .95. Subjects were asked to produce a pencil trace through all four points and to keep this movement going in a repetitive fashion. It was emphasized, however, that this was not an aiming task, and passing through each point was not necessary; subjects should merely try to pass close to each mark and maintain a stable pattern. Each subject was allowed to practice on identical sheets prior to testing. No cues were given as to the required speed of execution except that subjects were asked to complete the task at an "easy, comfortable pace." All subjects were right-handed and performed the ellipses in a counterclockwise direction. Although no indication was given as to the required limb movement, the spatial extent of the ellipses did not afford the possibility of completing them solely through wrist flexion and extension. As a result, all subjects automatically used a combination of wrist, elbow, and shoulder movement, with the hand held clear of the writing surface. On completion of all four ellipses, subjects were once again presented with each pattern and asked to complete the same task at approximately twice their previous speed. In the following analyses, the two speed conditions will be referred to as normal and fast speed. In the initial testing sessions (3 subjects) each subject continued to draw for approximately 10 s of which 4.25 s were recorded after the performance appeared to have stabilized (about 5–6 s). In latter sessions (3 other subjects) the recorded time was extended to 12 s. After the end of the recording period, subjects were instructed to stop.

Data Processing

Coordinate data were smoothed by using a double pass of a Butterworth second-order digital filter with a low-pass cutoff of 8 Hz. Derivatives of displacement data were calculated by using successive applications of a 5-point local polynomial approximation (Lanczos, 1957; Wann, 1987). The low-pass cutoff was chosen after inspection of both the second and third derivatives of sample data. Instantaneous curvature was calculated by using the first and second derivatives:

$$C = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{1.5}}. \tag{4}$$

Speed, perimeter, and eccentricity were calculated for each cycle of each trial, and the spatial variability was appraised. This measure took the x,y position of the spatial extremes along each axis for each cycle and then calculated a planar standard deviation for their dispersion. A coefficient of spatial accuracy was also calculated which expressed the standard deviation along a single axis, at each extreme, as a ratio of the total movement extent for that axis.

Results

Shape, Speed, Perimeter, and Spatial Variability

The geometric and temporal variables calculated for each subject's performance displayed a high degree of stability, given the lack of constraints upon their mode of execution. When measures were averaged across cycles, mean eccentricity of the ellipses was .68, .82, .92, .95, respectively, with no significant variation with speed or across subjects. The general trend toward elongating less eccentric patterns is interesting in that it may indicate a general trend by subjects to differentiate between circular motion and elliptical motion, resulting in overshooting with the more ambiguous ellipses. The average ellipse perimeter across subjects was stable around a mean of 32 cm, and the coefficient of variation was less than 3% under any combination of factors (subject, speed, ellipse). Average cyclic frequency of subjects for the normal speed condition was 0.94 Hz (approximately 30 cm/s) with 1.51 Hz (48 cm/s) under the faster speed condition, $F(1, 5) = 15.32$, $p < .025$. There was a significant trend toward slightly faster execution times with more eccentric ellipses, particularly for the normal speed condition, with average frequency varying from .85 Hz (.60 ellipse) to .97 Hz (.95 ellipse), $F(3, 15) = 3.81$, $p < .05$. Finally, there was no evidence of any differences across conditions in the spatial variability estimates, which appraised the tightness of the grouping at each of the vertical and horizontal extremes, across cycles. The planar spatial standard deviation (see Method) was generally of the order of 0.5 cm for the longer horizontal axis and 0.3 cm for the vertical axis.

Appraisal of a Power Law: Principles

In order to appraise the adequacy of a one-third power law, two levels of analysis are undertaken. The initial level adopts a straightforward approach of analyzing each movement cycle. The results of this analysis, however, are then used to highlight the potential for misinterpretation of results when analyzing data in this manner.

The appraisal of a power law may be attempted by using a least-squares regression procedure based upon Equation 3. A linear regression of the natural logarithms of V and R allows the estimation of the least squares exponent (β) for R. The standard error of the estimate of β (SE_β) may then be used to appraise the significance of any linear divergence from a one-third law, in the form of an F statistic:

$$F = \left(\frac{\beta - \frac{1}{3}}{SE_\beta}\right). \quad (5)$$

The data used for the regression should conform to the limits of Equation 2; hence, low curvature sections should be excluded. In this case, all radius values greater than 15 cm were rejected.

Two potential problems may be perceived with this approach.[1] First, successive values of R cannot be independent because of the continuity requirements of an ellipse. For this reason, the regression procedure randomly selected only 30% of the coordinate pairs for analysis. The rejection of a large portion of the data is unusual and may be viewed with suspicion by the reader. To allay any such reservations, the analyses were also checked on the full data set. All of the effects to be reported were also significant, at a similar level, under a full analysis, and all means were within a few percent of the reported value. The only noteworthy trend of the random rejection procedure was to reduce the correlation between β and ellipse perimeter from $-.2147$ (full analysis) to $-.0398$ (random).

The second issue that may have a bearing is that the data of Viviani and McCollum (1983) on circular drawing suggest there may be an effect of perimeter variations on the calculated exponent β. If perimeter varies grossly across cycles and if all cycles are included in the analysis, this may introduce error. In this experiment, however, as already reported, perimeter exhibited very minor variations. In addition, the protocol adopted was to regress each cycle of the ellipse (or segment of a cycle in later analyses) independently and to appraise the power law on each. In each regression, perimeter was therefore a constant and could not affect the validity of the estimates. Support for this last contention is borne out by the negligible correlation between β and the perimeter reported in the previous paragraph.

Table 1
Averaged Parameters for Velocity-Curvature Regression

Speed and ellipse	Exponent β	Gain K	Correlation[a]
Normal			
.60	.256	19.01	.800
.80	.296	18.36	.885
.90	.336	17.27	.970
.95	.342	15.90	.975
Fast			
.60	.324	26.70	.930
.80	.336	27.88	.975
.90	.331	26.34	.980
.95	.343	23.92	.990

[a] Correlations averaged using z transforms.

Appraisal of a Power Law: Initial Findings

All the trials analyzed could be well described by some form of power law between velocity and radius of curvature. The exponent which yielded the best fit, however, was not constant for all patterns and did not always agree with a ⅓ approximation. There was a significant linear effect of eccentricity, $F_{\text{lin}}(1, 15) = 29.38$, $p < .001$, and an interaction between eccentricity and speed of execution, $F(3, 15) = 5.65$, $p < .01$. Table 1 displays the relevant means along with other regression statistics. It can be observed that in the normal speed condition, the exponents gradually attain a ⅓ level with increasing eccentricity, whereas under the faster speed condition all exponents are of a similar order. The velocity gain

[1] We are particularly grateful to Paolo Viviani for his suggestions with respect to the problems underlying such an analysis.

factor K (Equation 2) exhibited a significant variation with speed, $F(1, 5) = 22.61$, $p < .01$, in line with the findings of previous studies, but also showed a strong linear effect of eccentricity $F_{lin}(1, 15) = 19.52$, $p < .001$. Finally, it should be noted that the correlation for the regression (averaged and analyzed using z transforms) showed similar effects for conditions of speed, $F(1, 5) = 45.31$, $p < .002$, and eccentricity, $F_{lin}(1, 15) = 371.5$, $p < .001$.

Thus it appears that under the condition where subjects chose their own rate, there were marked deviations from a one-third power law and that these were associated with the eccentricity of the ellipse. It addition, it is prudent to note that the direction of such deviations is opposite to that predicted if there had been an influence of perimeter variations (Viviani & McCollum, 1983).

Interesting though these effects may be in terms of modeling movement, the acid test of whether the ⅓ value is still stable enough to constitute a "law" is outlined in Equation 5. Did the least squares exponent differ significantly from ⅓? An F statistic was calculated for each regression of each cycle, and the number that reached significance ($p < .05$) was then totaled and expressed as a percentage of the total number of cycles. This approach avoids blurring any of the assumptions underlying such a test. The results are clear: Of all the exponents calculated (approximately 80 regressions per subject), only 7.6% were significantly different from ⅓ for the .60 ellipse, and none were significantly different for the other ellipses.

The simple conclusions from the preceding analyses would be that variations may occur in the power exponent for the velocity curvature relation but that a ⅓ approximation is stable enough to be treated as a definitive value; however, we ask the reader to reserve judgment on this issue until a brief analysis is detailed which explores the possible cause of deviations from ⅓. This will suggest that the foregoing level of analysis is not appropriate to resolve this issue, and therefore the use of the quoted statistics in support of the one-third law would be unwise.

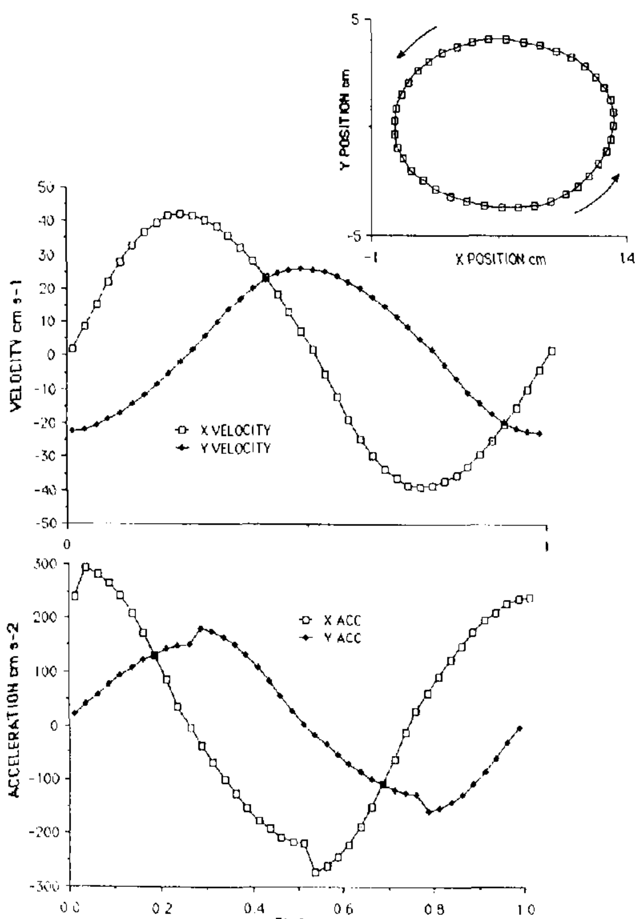

Figure 2. Simulated elliptical motion to demonstrate the effect of a 10% skew in sinusoidal velocity profiles. Upper panel: Velocity profiles are skewed so that the peak velocity occurs at .45 of the half cycle (zero crossing to zero crossing). Inset: The spatial pattern for the ellipse maintains a .80 eccentricity despite slight spatial distortions. Lower panel: The acceleration profiles for the x and y axes (X ACC, Y ACC) display a "fine-structure" around the peaks due to the nonsymmetrical zero-crossings of the velocity profiles.

Simulating Deviations From a Power Law

Given the findings of the previous section, it is interesting to consider what conditions are sufficient to model the deviations observed from a one-third law. In the introduction it was demonstrated that the combination of two sinusoidal or sine-like oscillations would produce a one-third law. The altering of phase between the two does not affect this prediction, other than to extend the range of curvature as phase approaches 0° or 180°. If we make the temporary assumption that a sine wave adequately approximates the velocity functions of human oscillatory movement (e.g., Soechting, et al., 1986; Soechting & Terzuolo, 1986), modulating amplitude, period, or phase of oscillation cannot explain such effects, particularly if we wish to maintain a constant spatial trajectory. Mild skewness,[2] however, can produce identical effects with very little distortion of the spatial trajectory.

Figures 2 and 3 illustrate the results of coupling two such sinusoids, both skewed by 10%, so the absolute peak values of velocity occur at .45 of the half cycle (time between two zero crossings of a velocity profile), rather than at .5. Future references to skew will use this value of the relative location of the peak, because it provides information about both

[2] Skewness in this context refers to asymmetry in the acceleration function either side of the peak velocity (acceleration zero-crossing) for a unimodal movement. The sinusoidal velocity functions used in this simulation were produced by the same approach that Maarse and Thomassen (in press) used:

$$v(t) = vm\sin(u),\ u = \frac{\pi}{2}\frac{t - t_0}{t_1 - t_0} \qquad t_0 < t < t_1$$

$$v(t) = vm\cos(u),\ u = \frac{\pi}{2}\frac{t - t_1}{t_2 - t_1} \qquad t_1 < t < t_2,$$

where vm = peak velocity, and t_0, t_1, t_2 are the times of the onset, peak, and cessation of motion, respectively.

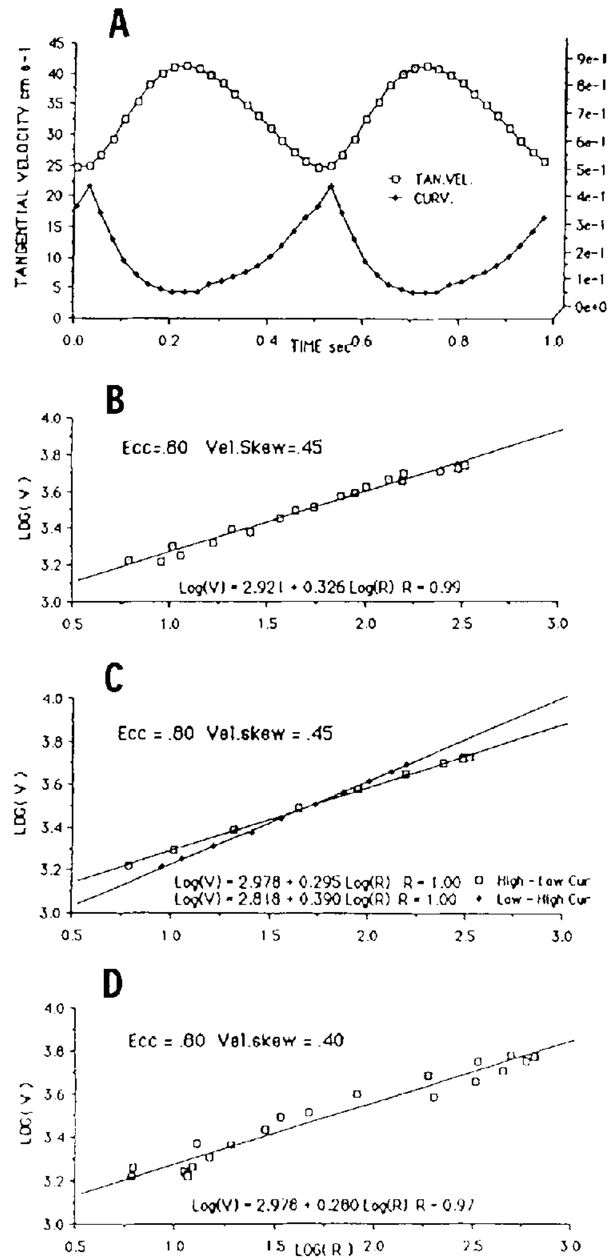

Figure 3. Simulated elliptical motion: The effects of skewness upon the velocity-curvature relation. Panel A: Velocity (TAN. VEL.) and curvature (CURV.) for a skew of .45 showing a lack of symmetry around points of high curvature. Panel B: log–log regression for a .45 velocity skew displaying a scattering of data points (compared with the lower panel of Figure 1) and a decrease from ⅓ for the least squares exponent. Panel C: Reanalysis of the data in Panel B to demonstrate that the asymmetry produces two power relations, which are different for motion towards a point of high curvature (Low–High Cur) and motion away from a point of high curvature (High–Low Cur). (These are averaged by the approach taken in Panel B.) Panel D: An increase in the skew to .40 results in wider dispersion of the data and a further decrease in the curvature exponent if a total cycle regression is used. (Ecc = ellipse eccentricity; Vel.skew. = proportional skewness of velocity profile.)

direction and extent of skew, and it is simple to estimate or compare across studies. The ellipse to be simulated was identical to that used for Figure 1 and performed by the subjects. It appears (inset Figure 2) that ellipse distortion is well within the bounds of normal human motor error.

The acceleration profiles show interesting discontinuities at their peaks, which at first glance seem to be incompatible with empirical data. Such "fine structure" in acceleration profiles of handwriting data, however, has already been commented on by other researchers (Dooijes, 1984; Hollerbach, 1980, 1981; MacDonald, 1966). In particular, Dooijes (1984) highlighted the unusual nature of such peaks in the second portion of the half-period, because they are precisely opposite to the effect that coulomb friction (pen–paper friction) would have on a symmetrical control function (see also Shinners, 1978). Hollerbach (1980) also doubted that such peaks were due purely to the influence of friction and suggested that at least some of the effects might be due to "segmentation of the acceleration profiles" (p. 55). This suggestion is essentially equivalent to the skewing procedure used in this simulation. In this respect, the simulation does not conflict with previous empirical findings, but provides an alternative perspective for their interpretation. Interestingly, it has also been demonstrated that the most effective models for the reconstruction of the spatial characteristics of handwriting are those that include similar skew effects in velocity functions (Maarse & Thomassen, in press; Maarse, van Galen, & Thomassen, in press).

The relation between curvature and tangential velocity for the skewed sinusoids is shown in Figure 3. At a skew of .45 the least-squares exponent for R begins to drop, (0.326). This is accompanied by a scattering of the data pairs although the correlation is still high (.99). The middle regression plot, however, demonstrates that this scatter is not random but is due to the fact that going out to any of the vertical or horizontal extremes of the ellipse is no longer the same as coming back. Regression lines calculated separately for the high to low and low to high curvature portion return to perfect correlation and demonstrate a pronounced departure from ⅓, toward both lesser and greater values. This effect is the result of correlating quarters of a sinusoidal cycle that, because of the skewing procedure, have slightly different periods. In a discrete simulation such as this, nonlinearity is not evident. The final plot of Figure 3 shows that as skew reaches .4, the exponent calculated from the total cycle continues to decrease. The scatter effect becomes very pronounced because of nonlinearity in the data (increasing difference in the ¼ cycle period). This suggests that if there is evidence of strong skewness, a power law for either segments or cycles, in truth, can no longer be assessed by linear (log–log) regression.

Appraisal of a Power Law: Principles Redefined

What the foregoing simulation has shown is that if there are skew effects present in the data, then (a) the least squares exponents calculated on a total cycle regression will deviate in the direction observed in the empirical data; (b) this calculated exponent will not be a true estimate but a trade-off between two separate segment power functions which deviate more strongly from 1/3; (c) the correlation between velocity and curvature will drop as observed empirically because the trade-off means that the total-cycle data are more variable. The consequence of these three factors is that if such effects are present, the initial analysis used to test the power law is not appropriate. The F statistic (Equation 5) not only compares a diluted deviation from the law, but also the standard error of its estimation is inflated by the very conditions that blur its difference, thereby negating any attempt to appraise the significance of departures from a 1/3 law. It follows, therefore, that a better approach is to regress quadrants of the ellipse between points of high and low curvature and to calculate an F statistic for each of these (Equation 5). It may also be prudent to point out that if no skewing is present, this approach should not affect the general finding of the previous (total cycle) analysis.

Appraisal of a Power Law: A Reanalysis

Direct evidence of skew effects is provided in a later section. Figure 4, however, presents some typical data for a single subject performing two extreme ellipses. There is some indication in such an example that the velocity curve for the .60 ellipse is less regular and symmetrical, while the log–log plot shows splaying around the average regression line, commensurate with the previous simulation. It is interesting to note that although values of log(R) above 2.7 were not used for the regression calculations (see section on *Appraisal of a Power Law: Principles*), the data plotted in Figure 4 do not show the marked nonlinearity found by Lacquaniti et al. (1983).

Each cycle of each ellipse was searched to determine the segmentation points at each of the horizontal and vertical spatial extremes; the regression procedure was then applied to each segment (see Figure 4 for segment labels). Once again the departure of the least squares exponent was appraised for each regression by use of its own standard error and degrees of freedom (Equation 5). The percentage of the total comparisons that were significant at the .05 level was then calculated. The pattern of significant departures was markedly different from the total cycle analysis, with 59.8%, 49.2%, 32.1%, and 19.1% of segments achieving significance for the .60, .80, .90, and .95 ellipses, respectively. This linear trend with eccentricity was significant, $F_{\text{lin}}(1, 15) = 75.28$, $p < .001$, and a mild effect of speed was evident, $F(1, 5) = 10.19$, $p < .05$. Figure 5 displays the least squares exponents averaged over subjects for each segment of each ellipse under both conditions. It may be observed that in line with the simulation, the exponents are both greater and less than one-third but tend to approach this value with increasing eccentricity and increasing speed.

On the strength of this analysis, the conclusions of the previous section may be reversed. Given the high proportion of significant departures from a one-third law under varying trajectory conditions, it must be concluded on the basis of these data that the law does not adequately describe any general principle of trajectory execution and is a suspect basis for any extended analysis technique.

The comparison of the present results with those of other studies using similar tasks is difficult. Both the experimental procedures and analysis approach may influence results. The general pattern of findings, however, does concur with some of the results presented (but not commented on) by Lacquaniti et al. (1983; Figure 4), where the power law deviated in a similar direction for slower ellipses. Given the marked different approach to analysis in the two studies, however, such an observation should not be given too much weight.

Some of the previous studies (Lacquaniti et al., 1983; Viviani & Cenzato, 1985; Viviani & McCollum, 1983) have used very small subject numbers (2–4) that typically include the investigators. It is not inconceivable that differing results not only may be due to imposed tempo effects and analysis routines but also may indicate a change of strategy with skill level.[3] Some support for this suggestion may be gleaned from the aforementioned studies. Viviani and McCollum (1983; Figure 1) depict simple circular motion that, in line with our data, displayed additional minor variations in local curvature, which are also typical of the fine-structure of our simulations (acceleration is used to calculate curvature and hence transmits transient variations). Viviani and Cenzato (1985; Figure 1), however, using the same data processing routines as the previous study, displayed elliptical movement that closely approximated harmonic motion, which surely suggests performance by a highly skilled subject. It is possible that increasing experience on such a relatively unusual task may lead the subject to adopt different strategies and criteria for control.

Discussion

The analyses presented in the previous sections have been aimed at establishing that human motion may transgress the one-third power law, which states that tangential velocity is proportional to radius of curvature raised to the power of 1/3. In this respect, the approach has been primarily destructive and has served to highlight that such a "law" may be a limit case approximation or may result from the adoption of inappropriate analysis techniques. It now seems appropriate to be more constructive and ask whether the observed deviations in the data serve to expand our understanding of control. To provide some perspective on this question, we have chosen to explore a model of control based upon the optimization of smoothness in movement.

[3] It is interesting to note, purely on the basis of subjective observation, that most of the subjects within this study did not find the task of drawing ellipses a simple motor act. Establishing the initial pattern of motion took some time (data were not recorded for this period), and minor deviations from an ellipse occurred frequently, requiring small trajectory corrections. All subjects were naive to the task, except for a single practice trial and a number of cycles that were performed prior to the start of data recording.

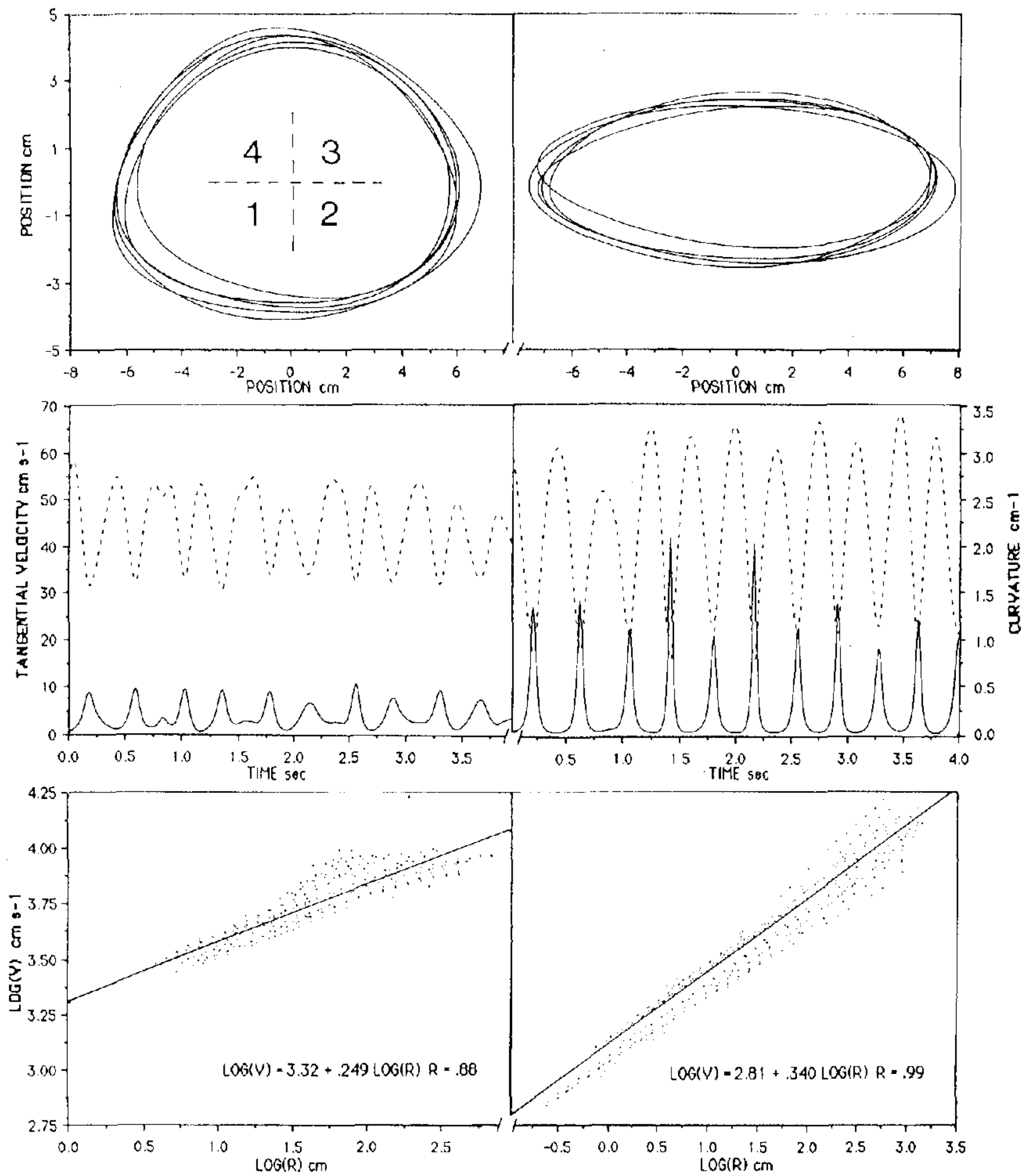

Figure 4. Empirical data from the performance of a .60 and .95 ellipse. Upper panel: Spatial patterns from 1 subject performing a .60 and .95 eccentricity ellipse, plotted on the same scale, (numbers inset on the .60 ellipse indicate the numerical labels used for each segment in the subsequent analysis). Middle panel: Plots of tangential velocity (dashed line) and curvature (solid line) for each respective ellipse, displayed on the same scale. (The average velocity is the same for both ellipses, but the range is greater for the more eccentric pattern, commensurate with the effects of curvature.) Lower panel: The natural log of tangential velocity plotted against the natural log of radius of curvature for both ellipses. (The ordinate is the same scale for both plots, but the abscissa is different. The solid lines indicate the average regression lines for each plot, the parameters of which are displayed in the inset equations.)

Extending the Notion of Minimum Jerk

The minimum-jerk model of Nelson (1983; adapted from the principles of Flash & Hogan, 1985) was presented in the introduction, and its prediction of the one-third power law was demonstrated. The notion of minimizing jerk, and hence optimizing smoothness, is appealing as a control model, and good agreement with empirical data has been demonstrated by Flash and Hogan for point-to-point movements. The unanswered questions pertaining to the role of such a model in the refinement and modulation of movement seem to be, Is there any equivalence between an individual's perception of "jerkiness" and mean-squared jerk as used by Flash and Hogan (1985)? If a child refined a spatial trajectory until it

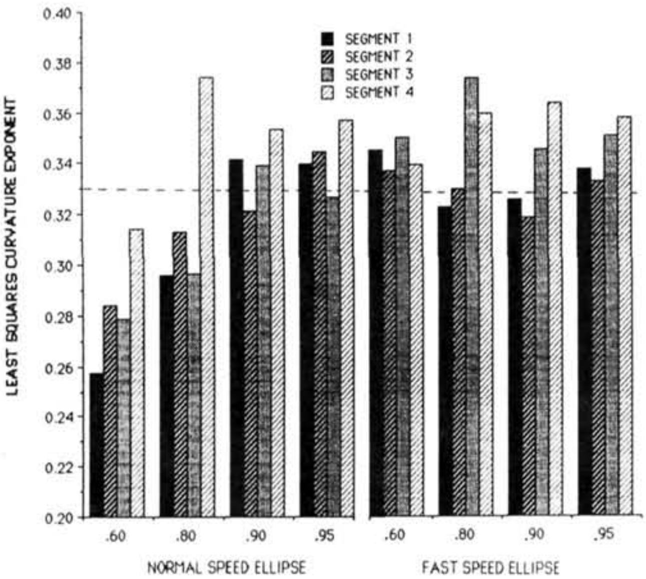

Figure 5. Least squares exponents for the velocity radius of curvature regression on empirical data, using a segmentation approach (see text). (Dotted line indicates a ⅓ law.)

"felt smooth," would that be equivalent to minimizing such a mathematical criterion? A simple observation pertinent to such questions is that we are not Newtonian point-masses or inextensible linkages but visco-elastic bodies. The criterion chosen for minimization by Flash and Hogan (1985) was based purely upon the trajectory jerk, that is the jerk "experienced" by a dense inextensible point-mass. Subjective perceptions of motion, however, seem to be conditioned by the physical state of human tissue. If someone is rapidly accelerated by a vehicle, elevator, or roller coaster, many of the sensations of acceleration are due to the distortion of their visco-elastic mass, and jerk is perceived similarly. Figure 6 presents a schematic of a visco-elastic mechanical jelly being driven through a trajectory. The black sphere represents a dense center of mass being accelerated or decelerated. If the system is able to monitor the strain in the surrounding tissue (springs), then the system is essentially a three-dimensional viscous accelerometer, blind to position, ambiguous about velocity, but aware of the extent, direction, and changes in acceleration.

We now propose to adopt these ideas to modify our concept of smoothness in movement. This model was originally developed with a different motivation, and a full discussion of the mathematical and psychological principles is presented elsewhere (Nimmo-Smith, 1988; Wann & Nimmo-Smith, 1988). Its adaptation to the current problem, however, seems both relevant and fruitful, and the principles of the approach are developed below, with a mathematical "skeleton" presented in Appendix A.

For a movement in one dimension, let the position of the center of mass be denoted by $x(t)$ and the distortion of the body or instantaneous stretch be denoted by $d(t)$. If it is assumed that the dynamics of the tissue can be adequately approximated by a linear second-order system, for a given amplitude and execution time, the redefined criterion for minimization becomes

$$\frac{1}{2}\int_0^T (\dddot{x} + \dddot{d})^2\, dt. \tag{6}$$

This is equivalent to minimizing both the external and internal jerk on the body. The mathematical solution to the optimization problem (Appendix A) is similar to that arrived at by Flash and Hogan (1985), in that it is based upon a quintic polynomial in time. However, it also includes exponential terms that are dependent upon the stiffness and viscosity of the system relative to its inertia. The predictions of such a model, in simple terms, are that if the system (limb tissue) is stiffened, then the optimal trajectory for point-to-point movement converges on the Flash and Hogan (1985) solution. If the limb is relaxed, however, the optimal trajectory has a velocity profile skewed toward the onset of movement, similar to the forms observed in many upper-limb movements (e.g., Atkeson & Hollerbach, 1985; Wing & Miller, 1984). There are limits to the applicability of the model in terms of human behavior; although there is little problem in specifying any trajectory for a system with high relative stiffness, attempts to drive a flaccid or extremely viscous system at high frequency produce optimal trajectories with pronounced

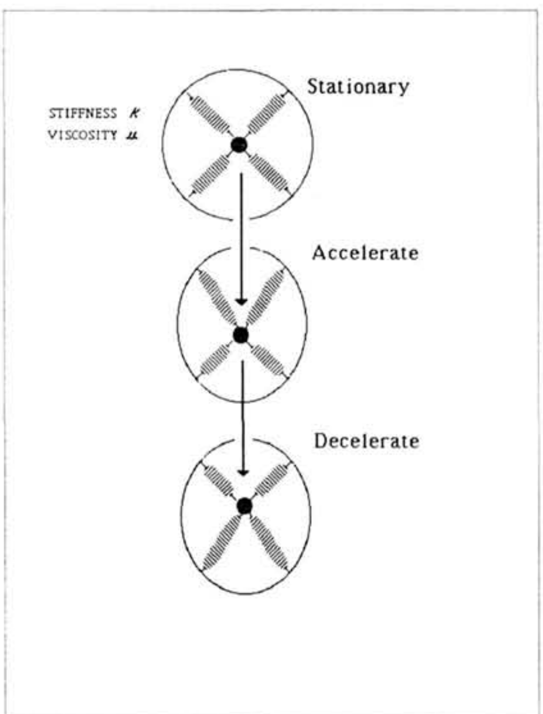

Figure 6. Schematic of a visco-elastic model for minimizing jerk. (The mass is driven through a trajectory based upon a dense center [black mass]. As the body accelerates or decelerates, small distortions in the surrounding tissue [springs] provide cues as to the direction, extent, and any changes in acceleration. The criterion to minimize is not merely the trajectory jerk but the disturbance of the body.)

overshoots or reversals. When adapted to the less constrained problem of repetitive motion with nonzero endpoint accelerations, such as the motions analyzed in this article, equivalent behavior can be simulated. A stiff system converges on the Nelson (1983) solution (Figure 1), whereas less stiff systems produce progressively skewed velocity profiles.

Jerk Costs for a Visco-Elastic Body

In extending this model to human limb movement, it must be strongly emphasized that there is no suggestion that we are modeling the actual physical characteristics of a limb. The model is truly a psychological rather than biomechanical one. Its principles are that what people perceive in terms of kinematic smoothness is conditioned by the state of their visco-elastic mass. The perceptual function is considered as a global kinesthetic one (as subjectively perceived), without postulating any single or combined afferent groups. It is apparent, however, that the human limb is richly endowed with cutaneous and muscular afferents that are able to sense stretch and change in stretch, as required by the system displayed in Figure 6.

Given the behavior of such a system, it is obvious that the specification of stiffness and viscosity should have some rational basis. It should be emphasized that the stiffness and viscosity of the limb tissue (κ, μ) are not synonymous with the vector impedance of the limb with respect to directional motion (Vincken & Denier van der Gon, 1985). We suggest, however, that the two are related in that a considerable contribution to the tissue stiffness must be from muscular contraction; hence, it would be difficult to increase limb impedance without increasing tissue stiffness. The converse does not necessarily hold because it may be possible to stiffen limb tissue through the action of orthogonal or minor muscle groups without dramatically altering limb impedance with respect to some vector motion. For the sake of this analysis, we will assume that the damping coefficient (α in Appendix A) is relatively constant at 0.5, chosen on the general observation that human tissue is certainly damped but that it is still possible to set up vibrations within the tissue. It may be observed from the simulation results depicted in Figure 7 that jerk cost (mean squared jerk) for a movement of given amplitude and duration decays as a result of increasing stiffness (undamped natural frequency). The form of this function scales with distance and time. The curve displayed is actually the total jerk cost for a .80 ellipse of the size used in previous simulations and performed by subjects. Because of the simple geometry of ellipses, however, the curve for other ellipses of the same perimeter would be identical. Given solely this criterion, it appears that the stiffer the limb tissue, the better, in that optimal (minimal-jerk) performance can be achieved only with a very stiff system that approximates a Newtonian point-mass. It is suggested, however, that purely on energetic criteria such a control strategy is not desirable. An increase in tissue stiffness can be achieved only at the expense of additional energy costs.[4] An alternative seems to be the selection of a stiffness level that substantially reduces jerk (i.e., to where the jerk cost begins to level out; Figure 7) without invoking excessive levels of stiffness. This conclusion may be reached

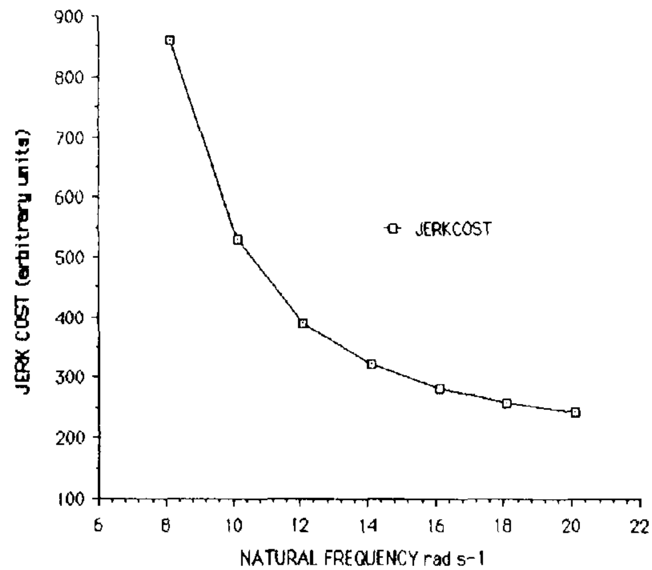

Figure 7. Simulated elliptical motion: Redefined jerk cost of performing a .80 ellipse as stiffness increases, plotted against undamped natural frequency. (Mean-square jerk for the visco-elastic model still retains normal spatial/temporal units, but these are arbitrary because they combine internal and external effects.)

without subscribing to any specific energy or effort criteria. Provided that one accepts that extreme tissue stiffness is undesirable, the exponentially decaying jerk cost illustrated in Figure 7 leads toward such a logical alternative.

Moving a Visco-Elastic Body Around Ellipses

The behavior of the visco-elastic minimum-jerk model under different stiffness conditions is shown in Figures 8 and 9. Figure 8 displays the simulated performance of a .80 ellipse at the same size and speed as used for previous analyses, with a tissue stiffness (undamped natural frequency, ω_0 in Appendix) of 12 rad/s. Spatial distortion of the ellipse is of the same order as Figure 4 and hence within the range exhibited by human performance. Velocity and acceleration profiles are qualitatively similar to those of the sinusoidal simulations, in terms of their respective skewness and fine-structure. These features arise from the inherent asymmetry of the cyclic kinematics for such a model. Such features may be lost during the smoothing of empirical displacement data, and it should be acknowledged that the behavior of this idealized trajectory model would be blurred if motion was generated through a second-order (muscular) system. As was discussed in the previous sinusoidal simulation, however, this fine structure

[4] The estimation of such costs in terms of metabolic requirements is notoriously difficult (e.g., Hatze & Buys, 1977). Nelson (1983) used a simple approach to the estimation of minimum-energy movement based upon mean squared acceleration (i.e., force per unit mass). This is clearly not appropriate because the same "energy" cost is estimated when the subject stiffens the limb and moves a set distance in a given time as when he or she relaxes and performs the same task (e.g., see Vincken & Denier van der Gon, 1985).

evident within the acceleration profiles is not at variance with empirical data and has been commented on by previous researchers (Dooijes, 1984; Hollerbach, 1980, 1981; MacDonald, 1966). It is also evident in the data of additional studies examining differing effector systems, (Edelman & Flash, 1987, Figures 3, 5; Kay, Munhall, Bateson, & Kelso, 1985, Figure 2), which reinforces the suggestion by Hollerbach (1980) that these are not friction artifacts but may reflect aspects of control.

Figure 9 illustrates the velocity-curvature relation for a .80 and .95 ellipse produced by the model using the same natural frequency (12 rad/s). It can be observed that the least-squares curvature exponent for the .80 ellipse deviates in a similar way from the empirical data (Figure 9C). As the relative contribution of motion along the minor axis decreases in the .95 ellipse, however, the exponent moves back toward 1/3, (Figure 9D). There is no need to change the natural frequency of the system to predict the effects observed in the empirical data. Figure 9E illustrates that if the system is stiffened (14 rad/s), then the least squares exponent for radius of curvature moves toward 1/3, but this is still the result of a trade-off between two markedly different values for each quadrant (Figure 9F; see earlier section on *Simulating Deviations From a Power Law* for a fuller discussion of this effect). This final demonstration is particularly germane to the observed trend in the empirical data, where an increase in speed of execution reduced deviations from a 1/3 exponent. If we accept that increased movement speed is accompanied by a tuning up of limb impedance (Vincken & Denier van der Gon, 1985) and a concomitant rise in tissue stiffness, then this model predicts convergence of the curvature exponents upon a value of 1/3.

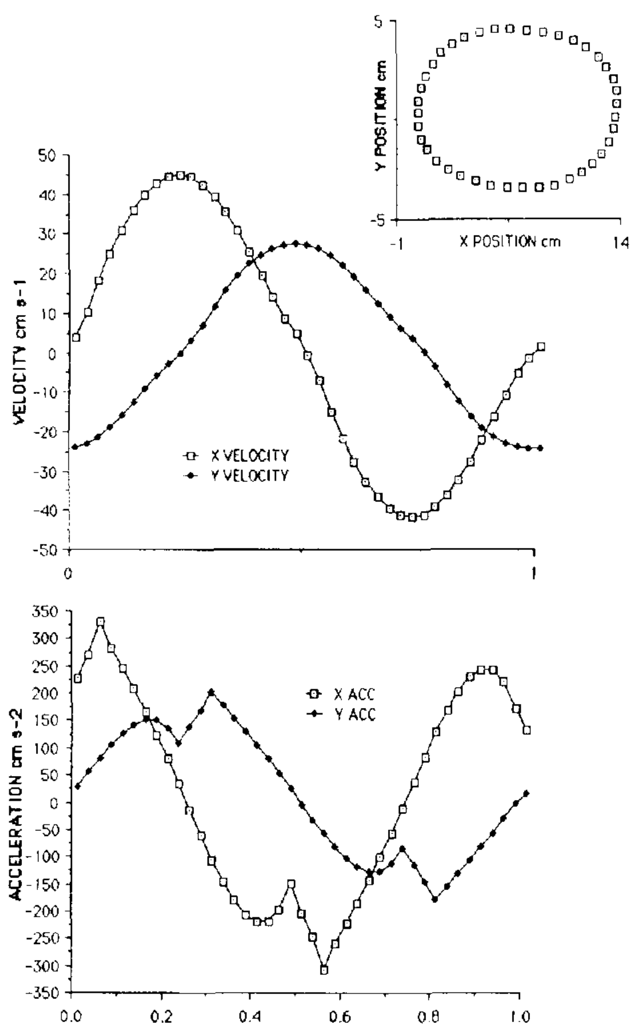

Figure 8. Simulated elliptical motion: Behavior of the visco-elastic model performing a .80 ellipse at 1 Hz. Upper panel: Velocity profiles showing slight skewness. Inset: Spatial pattern retains the general form of a .80 ellipse. Lower panel: Acceleration profiles for both axes (X ACC, Y ACC) exhibiting the "fine structure" typical of earlier simulations. (Compare with Figure 2.)

Appraising the Merit of a Visco-Elastic Model for Jerk Minimization

The behavior of the model presented in the previous sections is entirely commensurate with the empirical data. The least squares exponent deviates from 1/3 in a similar way for low-eccentricity ellipses, and there is evidence that this is the result of a trade-off between different quadrants of the elliptical trajectory. In order to produce these effects, it was proposed that the trajectory be optimized to minimize jerk at a given level of tissue stiffness but that the level of stiffness chosen be one that will substantially reduce potential jerk cost without unduly tapping the resources of the muscular system (i.e., not invoking maximum tissue stiffness). Given the latitude of such a model, it is possible to chose conditions that produce results in quantitative agreement with the empirical data, but such an approach may be considered little better than curve fitting the velocity and curvature records. There is obviously some question, therefore, as to whether the model, in common with many others previously proposed, merely mimics the empirical results rather than explains them. There is a need to examine independent predictions of the model with respect to the data. The model presented predicts that the observed effects are due to asymmetries in the velocity profiles, so that peak velocity is reached before the midpoint in movement time. This feature of the model can not be changed by any lawful manipulation of the parameters and provides a potential means of partial validation: It was demonstrated in earlier sinusoidal simulations (Figures 2 & 3) that skewness was a sufficient, though not necessarily the only, condition to produce deviations of the type observed in the data. The direction of the skew necessary to produce deviations, however, is not predetermined. Therefore, an initial appraisal of the adequacy of the model can be made as to whether skew effects are in fact present in the data and whether the direction of skew is in line with the model predictions.

A post hoc analysis of the empirical data was performed which searched the x,y velocity profiles to ascertain the temporal location of the peak for each trajectory. It should be emphasized that the model was developed independently and

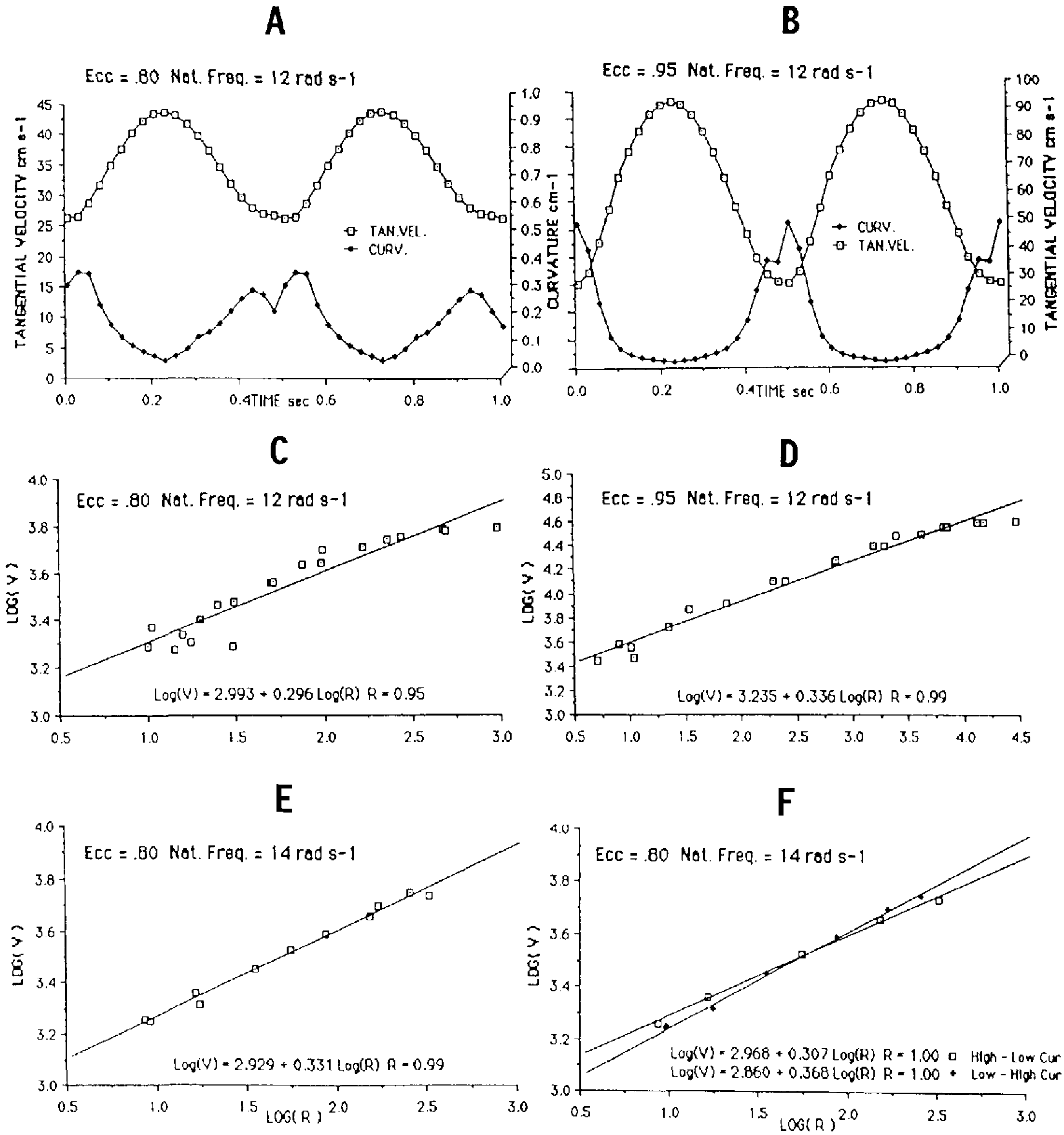

Figure 9. Simulated elliptical motion: Velocity curvature relation for the visco elastic model. Panels A and B: Velocity (TAN. VEL.) and curvature (CURV.) records for simulations of a .80 and .95 ellipse respectively, with a stiffness parameter (undamped natural frequency) of 12 rad/s. Panels C and D: Velocity upon radius of curvature regressions for a .80 and .95 ellipse respectively. (Note the variation in least-squares curvature exponent despite the same natural frequency for both simulations.) Panels E and F: A .80 ellipse simulated with 14 rad/s undamped natural frequency for the model. (The regression approaches a ⅓ power law in E. If the same data are reanalyzed in terms of segments, however [see section on *Appraisal of a Power Law: Principles Redefined*], the effects demonstrated in Panel C of Figure 3 are still evident in Panel F. Ecc = ellipse eccentricity; Nat.Freq. = undamped natural frequency for the model.)

without knowledge of any skewness within the present data. Thus the analysis of skewness was truly post hoc with respect to such predictions. The model predicts that skewing should be evident in the velocity profiles so that the velocity peak should occur at less than .5 of the movement duration. This effect should be primarily evident only for the slower (normal speed) trials, whereas faster trials should be more symmetrical. The relative peak locations for fast and normal speed trials over the four ellipse conditions are shown in Table 2. The results are generally in line with the model with a significant effect speed of execution, $F(1, 5) = 7.98$, $p < .05$. There is, however, additional evidence of a mild trend toward less skewness in more eccentric ellipses, $F(3, 15) = 3.17$, $p = .055$. This does not go against the model but merely suggests that there is a mild trend toward the use of greater stiffness for more eccentric ellipses, a suggestion in line with the earlier observation of a slight increase in execution speed with increasing eccentricity. One anomaly, however, is that the velocity peak for the ellipses at the faster speed generally occurred after the temporal midpoint (Table 2). This cannot be accounted for by the present model, which becomes perfectly symmetrical with increasing stiffness. The resulting conclusions must be that the data seem to provide qualitative support for the present model under relaxed conditions where subjects chose their own movement tempo. Where subjects are required to move somewhat faster, however, factors other than those accounted for in the present model seem to influence the trajectory dynamics.

Summary and Conclusion

The stated aim of this investigation was twofold. First, the stability of the one-third law was examined. Second, the relevance of any deviations in the velocity–curvature relation to our understanding of control was assessed. The results arising from the former analyses indicated that the one-third law may be a good approximation for movement performed at faster tempos. If constraints are relaxed and subjects perform movements at their chosen rate, however, there are significant deviations from such a law. Because the latter mode of performance is more akin to everyday movement, it must be concluded that the one-third law is an approximation possibly limited to imposed tempo movement (as used in many of the previous studies). A second important finding was that the approach used to analyze cyclical movements may bias results toward a one-third value, whereas the appropriate velocity–curvature power law along specific segments may be markedly different. This study compared approaches of taking individual cycles for the regression procedure as opposed to segmenting each cycle. Because previous studies have typically averaged data over large portions of a cycle and over cycles (Lacquaniti et al., 1983; Viviani & McCollum, 1983), their one-third finding is not surprising.

The variation of tangential velocity with curvature does seem to be a general principle in movement organization and has been observed with children as young as 5 months (Fetters & Todd, 1987). Thus it can provide a powerful focus for examining possible means of human motor control. We suggest that changes in the value of a velocity–curvature power law with experience or during development may be a particularly insightful area for future research, provided that appropriate analysis procedures are adopted. The one-third power law, however, seems to be a limit case approximation, and as such its use in the construction of analysis approaches or for the interpretation of data seems unwise. From the approach of modeling movement control, however, the general finding that a velocity–curvature power relation is normally present and in the range of 0.25–0.4 suggests that 1/3 is a good first approximation of the general trend.

A particularly interesting feature of the one-third law as it stood was the identity it held with coupled sinusoids. We have argued, however, that sinusoidal motion does not on its own constitute a biological control theory. A number of other control models produce movements that approximate sinusoidal motion, and a one-third law lent support to such control hypotheses. From suitable candidates such as weak nonlinear limit-cycle oscillators (Kay et al., 1987), minimum-energy (Nelson, 1983), best stiffness (Hasan, 1986), and minimum-jerk motion (Flash & Hogan, 1985; Nelson, 1983), we chose the latter for investigation. We found that the deviations from one third, observed within the empirical data, were commensurate with a model of minimum-jerk that took some account of the visco-elastic nature of human tissue and its possible consequences for the perception of jerk. It seems appropriate that such a model provides a good account of human behavior where subjects were performing under relaxed conditions. In such circumstances, it might be expected that both smoothness and effort of movement might be critical factors, whereas when additional temporal or spatial constraints are imposed, performance may be dictated more by such concerns. Hence, it may be concluded that the present model is a useful extension of the original notion of minimum jerk, in that it provides some potential for experiential learning of trajectory control. The model presented in this article, however, is still at a nascent stage, and thus we reserve judgement as to its validity across different tasks and conditions, pending further experimentation. On the basis of the data analyzed in this study, it appears to be limited to relatively unconstrained movement. The precise limits of applicability require further investigation, as does the validity of the model across conditions where stiffness is consciously modulated. On the strength of the present investigation, however, the behavior of a visco-elastic model for the minimization of jerk seems to provide an interesting area for further investigation.

Table 2
Averaged Skewness of Velocity Profiles (Time of Peak Velocity/Duration of Movement)

	Ellipse			
Speed	.60	.80	.90	.95
Normal	.475	.472	.490	.495
Fast	.508	.507	.520	.521

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Appendix A

A Visco-Elastic Model for Jerk Minimization

The visco-elastic model of trajectory formation is based on the proposition that when the system moves along a trajectory $x(t)$, it induces an internally represented deformation $d(t)$. Defining κ and μ as the ratios of stiffness/inertia and viscosity/inertia, respectively, then

$$\kappa = \omega_0^2, \ \mu = 2\alpha\omega_0, \qquad \text{(A1)}$$

where ω_0 is the undamped natural frequency, and α the damping ratio for the system.

The internal and external consequences of motion are related by a second-order differential equation of the form

$$\ddot{x} + \ddot{d} = -\mu\dot{d} - \kappa d. \qquad \text{(A1)}$$

Let $p(t) = x(t) + d(t)$ represent the trajectory followed by the center of mass, which may be thought of as the "perceptual center" of the system. We propose the "minimum-perceptual jerk criterion" for trajectory formation. Choose $x(t)$ to minimize

$$\int_0^T (\dddot{p})^2 dt, \qquad \text{(A2)}$$

subject to boundary conditions on $x, \dot{x}, d, \dot{d}$, at $t = 0, T$. For repetitive motion these are chosen to be

$$x(0) = x_0$$

$$x(T) = x_f$$

$$\dot{x}(0) = \dot{x}(T) = 0$$

$$d(0) = -d(T)$$

$$\dot{d}(0) = \dot{d}(T) = 0.$$

Standard constrained variational methods lead to solutions of the form

$$x(t) = g_5(t) + ae^{\lambda t} + be^{-\lambda t}, \qquad \text{(A3)}$$

where $g_5(t)$ is a quintic polynomial, whose six coefficients, together with a and b, must be able to satisfy the end-point conditions. The exponential term is a ratio of stiffness to viscosity $\lambda = \frac{\kappa}{\mu}$; substituting from Equation A1 gives $\lambda = \frac{\omega_0}{2\alpha}$.

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