Angle between vectors: Inner Products

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Angle between vectors: Inner Products

A natural quantity is the angle between two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^N$ or C^N.

Example 1:

In $\mathbb{R}^2$:

Properties an angle should have

1. Angle should take any two vectors $\mathbf{x}$, $\mathbf{y} \in S$ and produce a real number, $\theta \in [0, 2\pi)$
2. Angle should not depend on the lengths (norms) of $\mathbf{x}$ and $\mathbf{y}$.

Exercise 1:

Derive from the first principles angle θ between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$:

Recollect: $(\| \mathbf{x} \|_2)^2 = \mathbf{x}^T \mathbf{x}$ is the lenght of $\mathbf{x}$
$\| \frac{\mathbf{x}}{\|\mathbf{x}\|_2} \|_2 = 1$ (**unit vector**)

- **Step 1** - Normalize $\mathbf{x}$ and $\mathbf{y}$ and draw:
- **Step 2** - Build 2 right triangles and difference vector $\mathbf{z}$, where

$$\mathbf{z} = \frac{\mathbf{y}}{\| \mathbf{y} \|_2} - \frac{\mathbf{x}}{\| \mathbf{x} \|_2}$$

NOTE: (1) $a^2 + b^2 = 1$, (2) $a + c = 1$, (3) $b^2 + c^2 = (\| \mathbf{z} \|_2)^2$, (4) $\cos(\theta) = a$

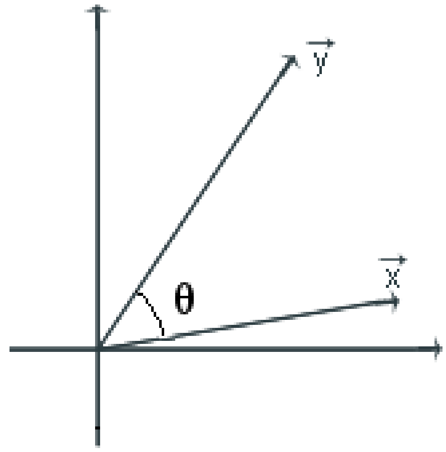

Figure 1: $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$

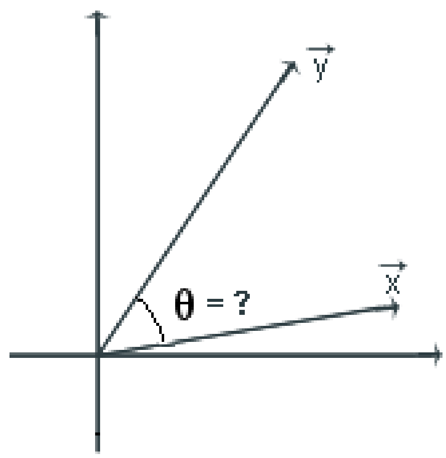

Figure 2

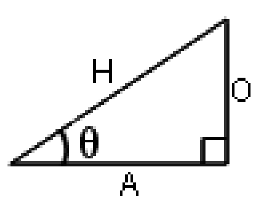

Figure 3: $\cos(\theta) = \frac{A}{H}$, $H^2 = A^2 + O^2$

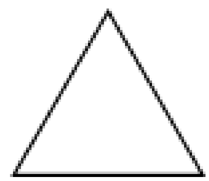

Figure 4: $\sum \text{angles} = 180$

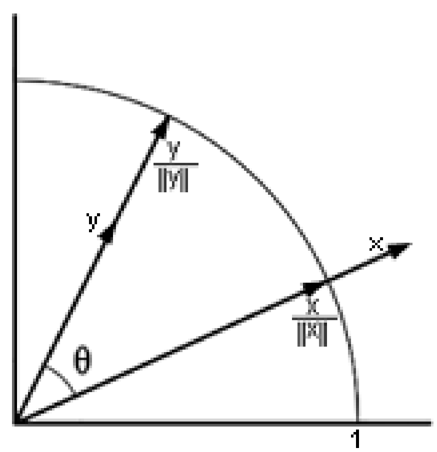

Figure 5

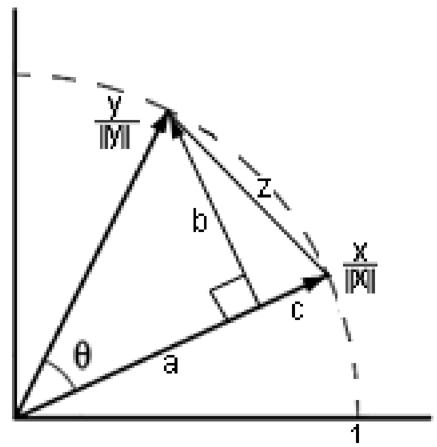

Figure 6

- **Step 3** - Pull it all together starting with (3), and using (1) and (4) for the last 2 steps respectively:

$$\begin{aligned}(\| \mathbf{z} \|_2)^2 &= b^2 + c^2 \\ &= b^2 + (1-a)^2 \\ &= b^2 + 1 + a^2 - 2a \\ &= 2 - 2a \\ &= 2 - 2\cos(\theta)\end{aligned} \quad (1)$$

therefore,

$$\cos(\theta) = \frac{2 - (\| \mathbf{z} \|_2)^2}{2}$$

but:

$$\begin{aligned}(\| \mathbf{z} \|_2)^2 &= \left(\| \frac{\mathbf{y}}{\|\mathbf{y}\|_2} - \frac{\mathbf{x}}{\|\mathbf{x}\|_2} \|_2\right)^2 \\ &= \left(\frac{\mathbf{y}}{\|\mathbf{y}\|_2} - \frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)^T \left(\frac{\mathbf{y}}{\|\mathbf{y}\|_2} - \frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right) \\ &= \frac{\mathbf{y}^T\mathbf{y}}{(\|\mathbf{y}\|_2)^2} - 2\frac{\mathbf{y}^T\mathbf{x}}{\|\mathbf{y}\|_2\|\mathbf{x}\|_2} + \frac{\mathbf{x}^T\mathbf{x}}{(\|\mathbf{x}\|_2)^2} \\ &= \frac{(\|\mathbf{y}\|_2)^2}{(\|\mathbf{y}\|_2)^2} - 2\frac{\mathbf{y}^T\mathbf{x}}{\|\mathbf{y}\|_2\|\mathbf{x}\|_2} + \frac{(\|\mathbf{x}\|_2)^2}{(\|\mathbf{x}\|_2)^2} \\ &= 1 - 2\frac{\mathbf{y}^T\mathbf{x}}{\|\mathbf{y}\|_2\|\mathbf{x}\|_2} + 1 \\ &= 2 - 2\frac{\mathbf{y}^T\mathbf{x}}{\|\mathbf{y}\|_2\|\mathbf{x}\|_2}\end{aligned} \quad (2)$$

therefore,

$$\begin{aligned}\cos(\theta) &= \frac{2-(\|\mathbf{z}\|_2)^2}{2} \\ &= \frac{2-\left(2-2\frac{\mathbf{y}^T\mathbf{x}}{\|\mathbf{y}\|_2\|\mathbf{x}\|_2}\right)}{2} \\ &= \frac{\mathbf{y}^T\mathbf{x}}{\|\mathbf{y}\|_2\|\mathbf{x}\|_2}\end{aligned} \quad (3)$$

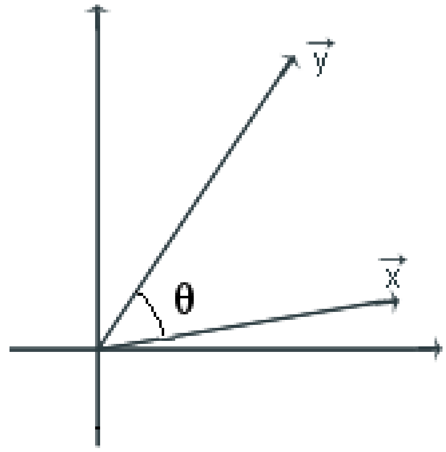

Figure 7

NOTE: $\theta = \arccos\left(\frac{\mathbf{y}^T\mathbf{x}}{\|\mathbf{y}\|_2\|\mathbf{x}\|_2}\right)$ is a real number in range $\theta \in [0, 2\pi)$ and is insensitive to lengths of $\mathbf{x}$ and $\mathbf{y}$.

Exercise 2:
Also, $\cos(\theta) = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2}$. Why?

1 More on Inner Products

$$\mathbf{x}^T\mathbf{y} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1y_1 + x_2y_2 = \sum_{i=1}^{2} (x_ix_i)$$

"Inner product", "dot product", "correlation coefficient".
So, angle $\theta \approx$ dot product.
θ and $\mathbf{x}^T\mathbf{y}$ measure "similarity".
$\mathbf{x}$ and $\mathbf{x}$ are in sense of direction.

Exercise 3:

How is inner product related to norm?

Inner Product: $\mathbf{x}^T\mathbf{y}$

Norm: $\mathbf{x}^T\mathbf{x} = (\| \mathbf{x} \|_2)^2$

2 Special Angles

$$\cos(\theta) = \frac{\mathbf{x}^T\mathbf{y}}{\| \mathbf{x} \|_2 \| \mathbf{y} \|_2}$$

- $\theta = 0 \Rightarrow \mathbf{x}^T\mathbf{y} =$

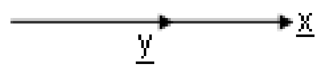

Figure 8: "colinear"

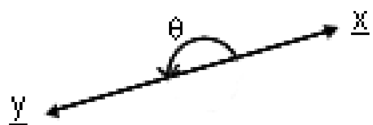

Figure 9: "colinear"

- $\theta = \pi \Rightarrow \mathbf{x}^T\mathbf{y} =$
- $\theta = \frac{\pi}{2} \Rightarrow \mathbf{x}^T\mathbf{y} =$

3 Cauchy-Schwarz Inequality for real time

Angle θ must lie between $-\pi$ and π. For example,

$$-\pi < \theta \leq \pi \Rightarrow -1 < \cos(\theta) \leq 1 \Rightarrow |\cos(\theta)| \leq 1$$

But

$$\cos(\theta) = \frac{\mathbf{x}^T\mathbf{y}}{\| \mathbf{x} \|_2 \| \mathbf{y} \|_2} \Rightarrow \frac{\mathbf{x}^T\mathbf{y}}{\| \mathbf{x} \|_2 \| \mathbf{y} \|_2} \leq 1$$

Therefore, the inner product is bounded:

$$|\mathbf{x}^T\mathbf{y}| \leq \| \mathbf{x} \|_2 \| \mathbf{y} \|_2$$

Exercise 4:

When do we have equality in CSI?

The Cauchy-Schwarz Inequality is fundamental in detection/classification problems of all kinds.

Example 2:

Idea: Compute $\forall i : |\mathbf{z}^T x_i|$ and choose the largest. Why? What is the maximum value the test statistics can take? What is the minimum?

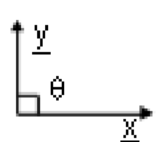

Figure 10: "orthogonal"

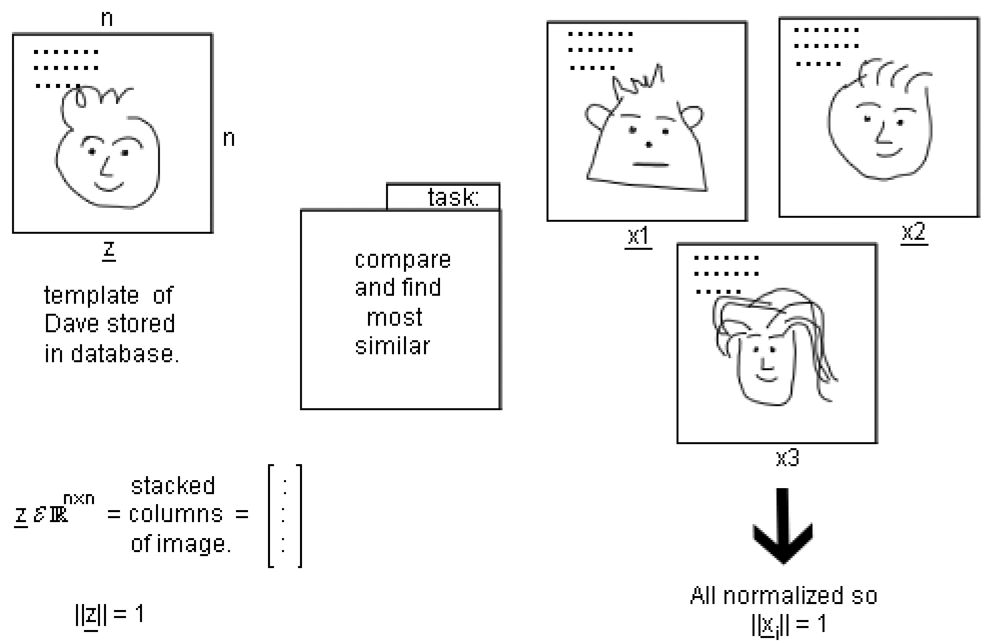

Figure 11: Face Recognition

3.1 Practical Issues

1. How close should γ be to 1? (threshold selection) What happens as $\gamma \to 1$? $\to 0$?
2. Registration - what if $x_4 = \mathbf{z}$ but shifted over a few pixels? How would you solve this? What would be the cost?

4 Properties of the Nth-Dimensional real inner product

The inner product can be denoted using different notations:

$$\begin{array}{rcl} \mathbf{x}^T\mathbf{y} & = & \sum_{i=0}^{N-1} (x_i y_i) \\ & = & (x|y) \\ & = & <x|y> \\ & = & <x,y> \\ & = & (x,y) \end{array} \quad (4)$$

The inner product takes each $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}$ into a 'number' $(x|y)$ such that:

1. $(x|y) = (y|x)$ (symmetry). Angle between $\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $\mathbf{y}$ and $\mathbf{x}$.
2. $(\alpha x|y) = \alpha (y|x)$ (scaling). Scaling $\mathbf{x}$ doesn't change the angle with $\mathbf{y}$.
3. $(x+z|y) = (x|y) + (z|y)$ (additivitity).
4. $\| \mathbf{x} \|_2 = (x|x) \geq 0$ and $(x|x) = 0$ only when $\mathbf{x} = 0$ (positivity).

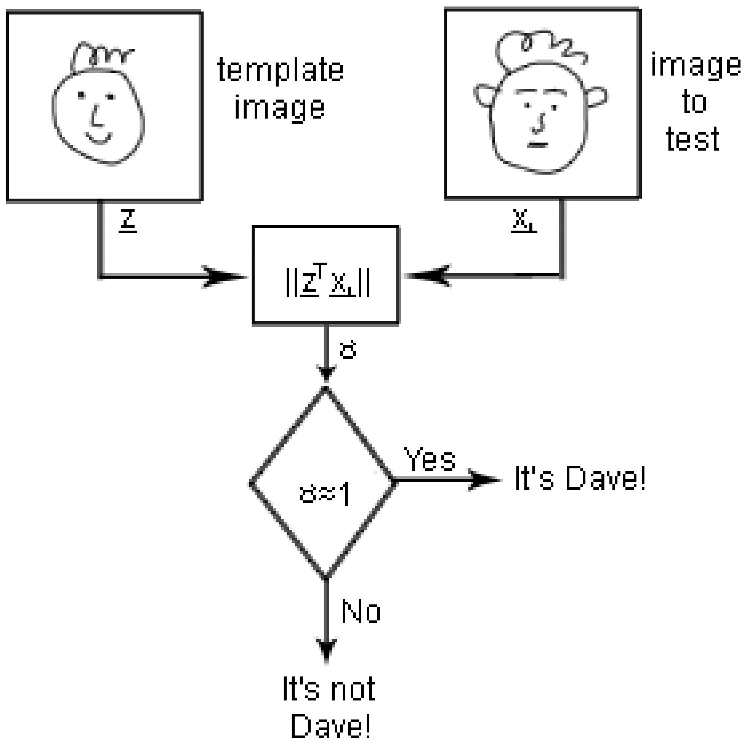

Figure 12: Recognition System

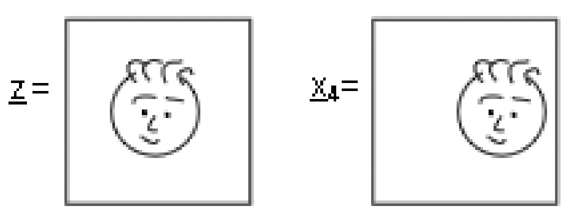

Figure 13

NOTE: 2 and 3 imply that the inner product is linear.

4.1 Generalization to Complex Domain

Angle between two complex vectors is

$$\begin{aligned}\cos(\theta) &= \frac{\overline{(\mathbf{x}^T)}\mathbf{y}}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2} \\ &= \frac{\mathbf{x}^H\mathbf{y}}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2}\end{aligned} \quad (5)$$

and so we define for C^2

$$(x|y) = \sum_{i=0}^{N-1}(x_i\overline{y_i}) = \mathbf{y}^H\mathbf{x}$$

Exercise 5:

Compute $(x|y)$ for $\mathbf{x} = \begin{pmatrix} 1+1i \\ 2+1i \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 0+3i \\ 1+2i \end{pmatrix}$.

Exercise 6:

How do the 4 properties of $\mathbb{R}^N$ inner product change for C^N?