

TRAJECTORY DETERMINES MOVEMENT DYNAMICS

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Abstract—The relation between figural and kinematic aspects of movement was studied in handwriting and drawing. It was found that, throughout the movement, the tangential velocity V is proportional to the radius of curvature r of the trajectory: $V = kr$, or, equivalently, that the angular velocity is constant: $d\alpha(t)/dt = k$. However, the constant k generally takes several distinct values during the movement, the changes being abrupt. These changes suggest a clear segmentation of the movement into units of action which overlap but do not coincide with the figural units as defined by the discontinuities of the movement (cusps, points of inflection). This organisational principle holds even when movements are mechanically constrained or are executed under strict visuo-motor guidance. Moreover, the segmentation of a given trajectory is invariant with respect to the total duration of the movement. A tentative interpretation of the principle is proposed which results from the assumption that the actual movement is produced as a continuous approximation to an intended movement, and that the well known relationship between movement speed and extent in rectilinear trajectories (Fitts' law) also applies to such continuous approximation.

The notion of movement encompasses two quite distinct aspects: the spatial description of the trajectory and the time evolution of the motion along this trajectory. *A priori* no relationship needs to exist between these two aspects. However, in the special case of linear (unidimensional) movements of the forearm and hand, some constraints have been shown to exist. Indeed, it has long been known^{2,15} that whenever a movement is not under careful visual guidance, the average velocity increases as an approximately linear function of the distance to be travelled, so as to keep the duration of the movement roughly constant (Isochrony Principle). A more general relation, known as Fitts' Law⁶, relates duration, extent and accuracy of the movement by stating that the information output of the human motor system is approximately constant. For a fixed value of the required accuracy, Fitts' Law states that movement time is a less than linearly (logarithmically) increasing function of distance. A similar compensatory effect is also found in saccadic eye movements where a negatively-accelerated relation (main sequence) exists between peak angular velocity and saccade amplitude.^{2,3} More generally, though, the problem of whether or not a relationship exists between spatial and kinematic aspects of movement can only be properly posed by considering movements which involve reversal of direction and curvilinear trajectories.* Handwriting is a prime example of such movements. However, most studies on this subject have been mainly concerned with

recording techniques^{3,9,12,24} and with the description of the trajectories obtained by fitting the kinematic parameters with suitable simple functions^{12,13,24} without addressing the problem stated above. Noticeable exceptions are Hollerbach's oscillatory theory¹⁰ and recent work of the Nijmegen group.^{17,18,20} In the work of Teulings and Thomassen angular velocity was measured, but the relevance of this parameter to the problem posed here was not recognized, possibly because of excessive filtering. In this report we will show that, in handwriting, a strict relationship exists between the form of the trajectory and the (tangential) velocity at which it is executed. Moreover, we will show that this relationship brings to the fore a segmentation of the movement into units of action. These conclusions also apply to other non-learned movements such as to the drawing of scribbles, as well as to the drawing of simple geometrical forms.

EXPERIMENTAL PROCEDURES

Movements were recorded with the help of a digitizing table (Calcomp, 622 RP) connected to an HP 21MX-E computer. The trajectory of the tip of the pen was sampled at 100 Hz with a position accuracy of 0.025 mm. The displacement of the pen could be reliably measured also when the tip was not touching the paper, as long as it was within 1 cm from the table. A binary z-axis variable indicated the written portions of the trajectory. Time derivatives of the coordinates $x(t)$ and $y(t)$ were calculated by the Lagrange polynomial method, after smoothing the raw data with a double-exponential, numerical low-pass filter (cut-off frequency: 50 Hz).

RESULTS

Figure 1 illustrates the basic finding with the help of two simple examples: an isolated letter (A) and a segment of scribble (B). In each case the two plots on

* Note: The term *Dynamics* refers to the time changes of forces and torques, while the term *Kinematics* refers to the time changes of position, velocity and acceleration. Throughout the paper the first term will be used to emphasize the motor plan which specifies forces and torques. The term Kinematics will instead be used to emphasize the geometrical parameters of the resulting trajectory.

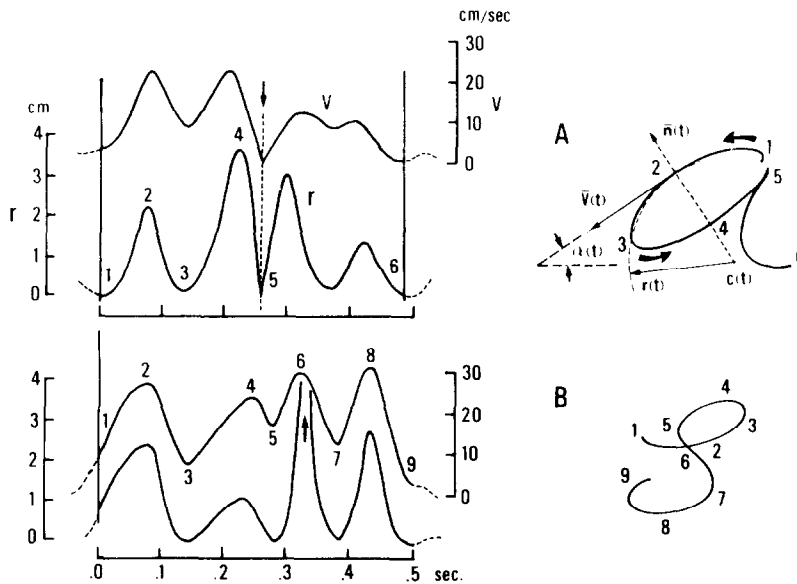


Fig. 1. Relationship between figural and kinematic aspects of handwriting and scribbling movements.

The instantaneous radius of curvature $r(t)$ and the modulus of the tangential velocity $V(t)$ completely describe the figural and kinematic aspects of the movement respectively, while the time course of the angle $\alpha(t)$ of the tangent to the trajectory resumes both these aspects. These three quantities are identified on the example of trajectory shown in A. The quantities r and V (left panels) were calculated from the instantaneous coordinates $x(t)$ and $y(t)$ recorded by the digitizing table. The relevant result shown in this Figure is the great similarity between the time course of r and V both in the case of a letter (A) and of a segment of extemporaneously generated scribble (B). Numbers on the trajectories permit to identify the corresponding kinematic events. Notice the presence of two singularities in the movement: the cusp (point 5 in A), where the tangential velocity goes to zero, and the point of inflection (point 6 in B) where the radius of curvature becomes infinite.

the left represent the time course of the radius of curvature of the trajectory (curve labelled r) and the modulus of the tangential velocity V (curve labelled V). The geometrical meaning of these quantities and that of the phase angle α are illustrated in A. Numbers provide the relationship between points along the trajectories and the corresponding values of $r(t)$ and $V(t)$.

The relevant finding illustrated by the figure is the striking similarity between the time evolution of r and V . This similarity was observed in all cases: isolated letters, words and scribbles. Thus, in the case of both learned, continuous movements and spontaneously-generated scribbles, a constraining principle exists whereby the figural and dynamical aspects of the movement are reciprocally related. This relation between the modulus of the tangential velocity and the radius of curvature is robust in the sense that it holds even when some types of external constraints are imposed on the movement (see later).

As a first approximation, the empirically-observed relation between $r(t)$ and $V(t)$ can be expressed as:

$$V(t) = kr(t) \quad a \leq t \leq b \quad (1)$$

where we admit that the proportionality constant k may depend on the interval (a, b) .

Standard calculations then lead to the following restatement of Eqn. 1:

$$d\alpha(t)/dt = k \quad a \leq t \leq b \quad (2)$$

where the phase angle $\alpha(t)$ is measured from an arbitrary horizontal reference (see Fig. 1, A). Thus, the constant k is the angular velocity of the movement over the interval (a, b) .

As explicitly indicated, the above analytical relation only applies piece-wise within non-overlapping time intervals. These intervals, however, remain to be specified. One intuitive way of dividing the motor action into segments identifiable under a variety of conditions is provided by two types of figural landmarks:

(1) *Cuspids*: where the direction of the movement is reversed as at point 5 in Fig. 1, A. At these points the radius of curvature is indeterminate and the linear velocity goes to zero.

(2) *Points of inflection*: where the sign of the curvature changes, as at point 6 in Fig. 1, B. At this point the radius of curvature is infinite while the angular velocity changes sign.

These landmarks represent points of discontinuity of the movement where the constant k appearing in

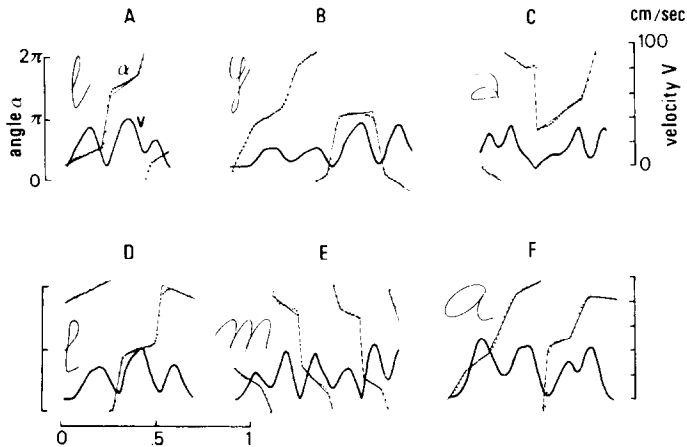


Fig. 2. Relation between spatial and kinematic aspects of movement for isolated letters.

Data from two subjects. In each panel the curves labelled V represent the modulus of the tangential velocity for the indicated letter. Dots (\cdot) show the values (module 2π) of the angle $\alpha(t)$ (see Fig. 1,A). The solid lines interpolating the data points are theoretical predictions derived from equations (1) and (2) in the text. Dashed lines correspond to points of discontinuity of the movement (as in C, E and F), or fast transitions for which no reliable prediction could be made (as in A and D). In all cases it is obvious that the slope of $\alpha(t)$ undergoes sharp changes which clearly identify discrete segments of the movement.

Eqn. (1) may be expected to change. Indeed, it is apparent from the example of Fig. 1, A that the average ratio, between tangential velocity and radius of curvature, is somewhat smaller for the segment of the letter at the left of the cuspid than for the subsequent segment. However, a finer segmentation of the movement emerges by looking directly at the time course of the phase angle. As shown in Fig. 2, F for the same letter of Fig. 1, A, the slope of a linear approximation to the phase angle (i.e. the ratio k) takes six distinct values (four before and two after the cuspid). These step changes in the value of the ratio k afford an additional criterion for segmenting the movement, which is based on the motion itself rather than on the resulting form.

In the other panels of Fig. 2, such analysis of the phase angle $\alpha(t)$ is carried out for other letters, while in Fig. 3 are shown the results for two simple geometrical forms which are of particular interest here. The circle is a limiting case in as much as, despite the large random fluctuations of the radius of curvature (see A), $\alpha(t)$ has a constant slope, i.e. the ratio k is practically constant throughout the movement. In other words, once a speed of execution is intentionally chosen, the motion consists of one segment only (compare A and B) in accordance—it would seem—with the fact that the ideal circle, as internally represented, has a constant radius. In the case of the ellipsis, instead, where the curvature varies considerably and systematically, step changes in the slope of $\alpha(t)$ clearly divide the movement into two pairs of symmetrical segments. Note also that the changes in the ratio k become more marked as the eccentricity of the curve is increased (compare C to D).

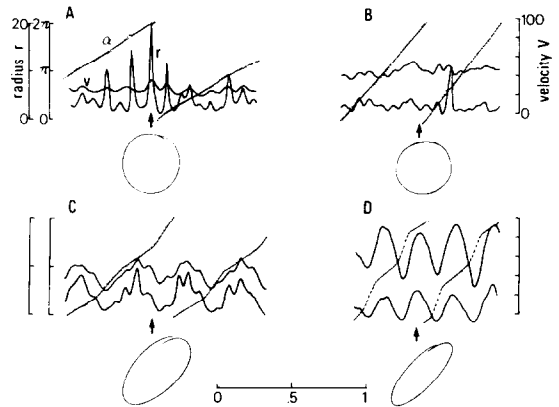


Fig. 3. Relation between spatial and kinematic aspects of movement for simple geometrical forms.

The significance of dots and interpolating lines is the same as in Fig. 2. To emphasize, in these particularly simple cases, the nature of the relation between spatial and kinematic aspects of movement, both the modulus of the tangential velocity (curve labelled V) and the radius of curvature (curve labelled r) are given. The fact that the slope of $\alpha(t)$ is piece-wise constant derives from the proportionality between V and r . Panels A and B show the results for two roughly equal circles drawn at different speeds. Panels C and D contrast the results for two ellipses with different eccentricities. In all cases the theoretical predictions, calculated as explained in the text, adequately fit each segment of the phase angle curve. In some instances (see panels A and C), the radius of curvature becomes very large, the corresponding segments of the trajectory being almost rectilinear. In these cases, the tangential velocity V is smaller than that necessary to keep the ratio V/r constant. As a result, the phase angle curve departs noticeably from a straight line.

Equation (1) is admittedly an approximation and so is the principle expressed by Eqn. (2). A verification of the limits of its validity can be obtained however by comparing the actual measurements of the phase angle $\alpha(t)$ with the theoretical line segments predicted by Eqn. (2). This comparison is shown in each example of Figs 2 and 3. In both Figures, the continuous lines which interpolate the data points have been drawn by computing the slopes (i.e. the constant k) as the *average* of the ratio $V(t)/r(t)$ over the corresponding time intervals. The location parameter is indeterminate because integration of equation 2 introduces an arbitrary constant. Therefore, this parameter was fitted by eye. The excellent agreement between predicted and measured values of $\alpha(t)$ demonstrates that the approximation expressed by Eqn. 1 is indeed very good.

As stated above, the relation between form and dynamics of the movement is also valid when some types of constraints are imposed. Figure 4 demonstrates this point in the case of mechanical constraints. It shows, with the same conventions as in Figs 3A, B, two examples of elliptic trajectories drawn by moving the pen along the inner rim of a plastic template. Obviously, the fact that the trajectory (but not the dynamics) of the movement is imposed does not prevent the establishment of a relation between the two.

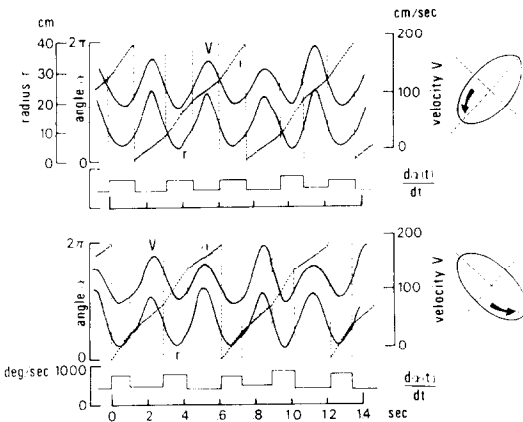


Fig. 4. Demonstration of the validity of the principle relating spatial and kinematic aspects of movement also in the presence of mechanical constraints.

The results, presented with the same conventions of Fig. 3, describe the drawing of an ellipsis obtained by moving the pen along the inner rim of a template. The upper and lower panels permit a comparison of the results obtained with the two indicated orientations of the template relative to the sagittal plane of the body. Aside from a phase shift, the dynamics of the movement is the same in both cases. The results observed in this mechanically constrained condition are very similar to those for the same ellipsis drawn freely (see Figs 3, C and D). The plots at the bottom of each panel permit a more direct comparison of the average slopes of the phase angle $\alpha(t)$ and illustrate the great regularity of their changes.

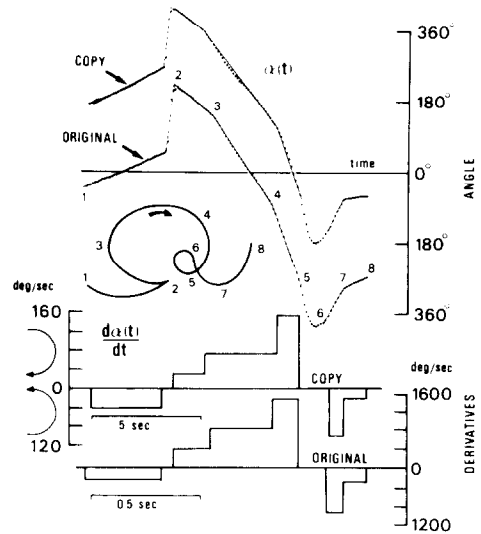


Fig. 5. Demonstration of the validity of the principle relating spatial and kinematic aspects of movement also in the case of visual guidance.

The two upper plots demonstrate the similarity of the changes in $\alpha(t)$, between an extemporaneously drawn scribble (curves labelled Original) and the same scribble copied by the same subject under careful visual guidance (curve labelled Copy). The two lower plots represent the slopes of the theoretical lines interpolating the phase angle curves $\alpha(t)$. Notice that the execution time for the copy was about ten times longer than for the original. The time scales for the two curves labelled Copy were compressed accordingly, to facilitate the comparison. The similarity between free and visually-guided drawing, extends even to the short time interval (from points 5 to 7) when the phase angle $\alpha(t)$ could not be reliably fit by straight lines.

Notice also that, aside from a phase shift, the results are independent of the main orientation of the movement with respect to the sagittal plane of the body (compare upper and lower curves), thus indicating that any reference to the slant¹⁰ is irrelevant to our problem.

Figure 5 further demonstrates the robustness of the principle by extending its validity to the case of movements accomplished under strict visual guidance. The specific task given to the subject was to copy accurately a scribble that he had previously drawn spontaneously. In spite of the fact that, in this particular instance, copying the model resulted in a tenfold increase of the movement duration (compare time scales), the evolution of the phase angle $\alpha(t)$ for the copy is virtually indistinguishable from that of the original, extemporaneously generated scribble. The slope of the linear approximation to $\alpha(t)$ for the two cases (lower plots) further dramatizes this similarity. Thus, the tangential velocity is proportional to the radius of curvature even when the trajectory is not spontaneously generated and the movement is accomplished under strict visual guidance. Notice also,

that the comparison shown in Fig. 5 represents an extreme example of homothetic behavior in the time domain. This behavior was originally demonstrated in typing,¹⁶ extended to handwriting by asking the subjects to write the same letter at different speeds, and further extended to the space-time domain by asking subjects to write letters of different sizes.²¹ In this respect one should note that the possibility of generalizing the Isochrony Principle to complex movements (see Introduction) had already been suggested by early chronometric studies^{1,7} and more recently by Katz,¹¹ Denier van der Gon & Thuring⁴ and Wing.²⁵

DISCUSSION

The experimental findings can be summarized as follows:

(1) The analysis of the relationship between the figural and kinematic aspects of continuous movements, either learned or generated extemporaneously, demonstrates the segmentation of the movement into well defined units of action.

(2) Within each of these units the figural aspects of the trajectory uniquely determine the velocity profile of the motion (see Eqn. 1).

(3) Within each unit of action the trajectory goes through equal angles in equal times (see Eqn. 2).

On the basis of point 3 above, we will refer to the principle relating form and dynamics as to the *Iso-gony Principle*.

The first point to be considered here is that of the most appropriate level at which the Iso-gony Principle should be discussed. To begin with, let us note that the data obtained when the movement is mechanically constrained may lead to believe that the Iso-gony Principle results from the bio-mechanical properties of the limbs, in conjunction with the general dynamical laws governing visco-elastic systems with many degrees of freedom. Indeed, most representations of handwriting emphasize the seemingly oscillatory nature of the movement. The basic assumption of these representations is that bidimensional trajectories result from the composition of two essentially harmonic, orthogonal vectors (see, for instance, Denier van der Gon & Thuring⁴). The intrinsically oscillatory nature of visco-elastic systems would then be responsible for the smoothness of the movement and reduce the necessity of a continuous central monitoring. Changes in shape, height and slant would be achieved by modulating either the forcing functions acting upon spring-mass-dashpot systems,^{12,22,24} or the setting of the visco-elastic parameters.¹⁰ Whatever the value of such formulations for describing the trajectories, it is easy to verify that the Iso-gony Principle cannot be satisfactorily derived within this framework. Even if we consider the simplest case, namely the drawing of an ellipsis, as in Fig. 3C, D, the oscillatory hypothesis would describe the

trajectory by the parametric equations:

$$x = A \sin \omega t$$

$$y = B \sin (\omega t + \phi)$$

whence:

$$\dot{\alpha}(t) = V(t)/r(t) = - \frac{AB \sin \phi}{A^2 \cos^2 \omega t + B^2 \cos^2 (\omega t + \phi)}$$

Instead, all our experiments have shown that the phase angle and its derivative $V(t)/r(t)$ can be adequately approximated by linear segments and step functions respectively. The same reasoning generalizes to more complex trajectories for which the parametric equations are represented by trigonometric series. Thus, while it can be readily assumed that the central control processes take into account bio-mechanical factors and in particular the visco-elastic properties of the muscles (and it is possible that they take advantage of these properties), continuous curvilinear movements cannot be satisfactorily parametrized using segments of harmonic functions, even for short time intervals.

Consequently, the Iso-gony Principle must pertain to the logic of the central control processes *per se* and its significance can only be discussed at this level. To begin with, the fact should be stressed that the results were identical when the subjects generated spontaneously a given trajectory and when the same trajectory was either copied under visual guidance (see Fig. 5) or drawn using a template (see Fig. 4). Thus, the Iso-gony Principle should not be inherent to the processes whereby the specification of the form is obtained either from an internal model or on the basis of externally imposed constraints. It should rather emerge within the implementation stage, a stage which we may logically distinguish from the process of retrieval of figural information from either memory storage or the external world. In the case of visually guided movements, this conclusion is somewhat surprising since several authors^{5,14,19} have chosen to emphasize the difference between spontaneous movement and movement under visuo-motor guidance.

As for the fact that the kinematics of free and mechanically constrained movements are the same (see Fig. 4), it may elicit the following question: in naturally unconstrained movements, is the control implemented by internally generating virtual boundaries (indistinguishable in their effects from those externally imposed) to be used as a reference for an appropriate utilisation of sensory afferences? A definite answer to this question must obviously await further experimental scrutiny.

Coming now to consider the Iso-gony Principle *per se*, the following interpretation can be proposed on the basis of two simple hypotheses. Let us assume that complex and continuous movements such as handwriting are performed by chaining a sequence of rectilinear segments (see Fig. 6).

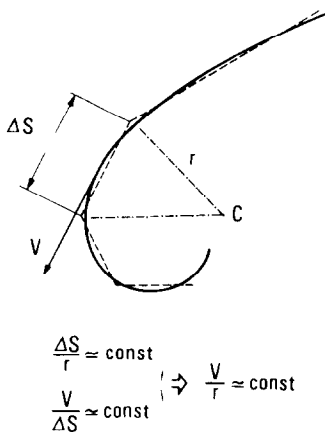


Fig. 6. Diagram illustrating the proposed interpretation of the Isogony Principle.

A curvilinear trajectory can be approximated by a sequence of line segments. In order to maintain the approximation error within a predetermined limit, the length of the segments must be made smaller when the radius of curvature decreases. If it is assumed that all segments are accomplished in equal times, the Isogony Principle can be readily derived.

The first hypothesis then states that transitions from one segment to the next occur whenever the mismatch between intended and actual trajectory exceeds a given threshold ϵ . Qualitatively, this implies that the higher the curvature, the shorter the approximating segments. More formally, the hypothesis states that the ratio between the length ΔS of the segment and the radius of curvature is approximately proportional to some function of the mismatch threshold:

$$\Delta S/r \simeq f(\epsilon) = \text{const.}$$

The second hypothesis states that all segments are accomplished in an approximately constant time, that is that the ratio between V and ΔS is again proportional to some function of ϵ :

$$V/\Delta S \simeq g(\epsilon) = \text{const.}$$

This is simply a restatement at the microlevel of the Isochrony Principle. The Isogony Principle immediately follows from these two premises:

$$(\Delta S/r) \times (V/\Delta S) = V/r \simeq \text{const.}$$

It is important to stress that we do not need to qualify how coarse the linear approximation is to the intended movement. In fact, under the mild condition that the product $f(\epsilon)g(\epsilon)$ remains finite when the maximum admissible mismatch tends to zero, the argument developed above stays valid even when the length of the segments become vanishingly small (continuous approximation). Thus, our interpretation of the Isogony Principle *should not* be taken to suggest that the movement is actually performed by a piecewise linear approximation to an ideal trajectory.

To conclude, we will briefly discuss the significance of this principle from the point of view of optimal control. The analysis of the angular velocity $d\alpha/dt$ has shown that continuous movements are segmented into units of action corresponding to successive portions of the trajectory. We have also argued that the Isogony Principle comes about in the implementation stage and indeed the interpretation we have just proposed is pertinent to such a level. Consequently, two processing stages may be conceptually distinguished: one in which the segmentation of the intended trajectory is planned, and one which supervises the execution of each successive unit of action in accordance with the principle. The first stage would not need to be concerned with a detailed dynamical specification of each unit of action. Aside from imposing the form and the size of the segment of trajectory to be executed, this stage could also control the overall speed of execution of the segment by simply setting the ratio $k = V/r$ to a proper value. In the second stage, instead, the actual velocity profile would be specified purely on the basis of the figural aspects of the trajectory. Thus, the Isogony Principle relating form and dynamics of the movement could be interpreted as evidence of an efficient organisation of the control processes.

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